

Lectures  
on the  
THEORY OF DISTRIBUTIONS

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by  
LEOPOLDO NACHBIN

Notes prepared by  
STELIOS NEGREPONTIS

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§28. Convolution of Distributions by Distributions

Let  $E$  be a real vector space of finite dimension  $n$ .

Definition 1. If  $\varphi \in \mathcal{B}(E)$ ,  $\psi \in \mathcal{K}(E)$  then  $\varphi * \psi \in \mathcal{B}(E)$  with respect to a fixed Haar measure, and  $\int (\varphi * \psi)(x) f(x) dx = \iint \varphi(t) \psi(x-t) f(x) dx = \int \varphi(t) \int \psi(x-t) f(x) dx dt = \int \varphi(t) (\hat{\psi} * f)(t) dt$ , for  $f \in \mathcal{K}(E)$ . Thus  $\widehat{\varphi * \psi}(f) = \hat{\varphi}(\hat{\psi} * f) = \hat{\varphi}(\hat{\psi} * f)$ . This will serve as motivation for the following definition. If  $S \in \mathcal{D}'(E)$ ,  $T \in \mathcal{D}'_c(E)$ , we define  $S * T \in \mathcal{D}'(E)$  by  $S * T(f) = S(\hat{T} * f)$  for  $f \in \mathcal{D}(E)$ . Note that  $\hat{T} \in \mathcal{D}'_c(E)$ , hence that  $\hat{T} * f \in \mathcal{D}(E)$ , by corollary 2, §27, and thus  $S * T(f)$  is well-defined. We must only show that the linear form  $f \mapsto S * T(f)$  is a distribution. This will follow by lemmas 1, 2, 3 below. We further note that the mapping  $(S, T) \in \mathcal{D}'(E) \times \mathcal{D}'_c(E) \rightarrow S * T \in \mathcal{D}'(E)$  is bilinear; and the definition of  $S * T$  does not depend on the choice of the Haar measure on  $E$ .

Lemma 1. Let  $T \in \mathcal{D}'_c(E)$  be fixed (definition 4, §22) and let by  $T$  the restriction  $T|_{\mathcal{D}(E)} \in \mathcal{D}'_c(E)$  as well. Then the mapping  $f \in \mathcal{D}(E) \rightarrow T * f \in \mathcal{D}'(E)$  is linear and continuous.

Proof: Linearity is clear. By proposition 1, §27, we have  $\frac{\partial^m (T * f)}{\partial x_1 \dots \partial x_m}(x) = T(t) \left[ \frac{\partial^m f}{\partial x_1 \dots \partial x_m}(x-t) \right]$ , by which continuity is clear.  $\square$

Lemma 2. If  $T \in \mathcal{D}'_c(E)$  is fixed, then the linear mapping  $\mathcal{D}(E) \rightarrow \mathcal{D}'(E)$  maps each  $\mathcal{D}_K(E)$ , where  $K \subset E$  is compact, continuously into  $\mathcal{D}_{K+K}(E)$ , where the spaces  $\mathcal{D}_K(E)$  have their natural topologies (induced by  $\mathcal{D}(E)$ ).

Proof: By proposition 3, §27,  $\mathcal{D}_K(E)$  is mapped into  $\mathcal{D}_{K+K}(E)$ . By lemma 1 and proposition 4, §22 continuity follows.  $\square$

Lemma 3.  $S \in \mathcal{D}'(E)$ ,  $T \in \mathcal{D}'_c(E) \Rightarrow S * T \in \mathcal{D}'(E)$ .

Proof: Apply lemma 2.  $\square$

Definition 2. If  $S \in \mathcal{D}'_c(E)$ ,  $T \in \mathcal{D}'(E)$  we define  $S * T \in \mathcal{D}'(E)$

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