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INTEGRALS AND ORDERS OF GROWTH OF DISTRIBUTIONS

by

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INTRODUCTION

In several papers dating from 1954 (see [8], [9], [10], [11] and [12] in References), I have tried to emphasize the advantages of an axiomatic approach to the theory of distributions, in connection with an idea of Bochner [1]. As is well known, one of the theorems given by L. Schwartz, starting from his definition of «distribution», states that every distribution T on an open set Ω in $\mathbf{R}^n$ can be represented, on each compact interval I contained in Ω, as a derivative $D^r F$ of a continuous function on I, where r is some system of n integers (F and r depending on I). If there is a system r and a continuous function F such that $T=D^r F$, the distribution T is said to be of «finite order».

The axiomatic characterization of distributions of finite order on an interval I in $\mathbf{R}^n$, in terms of «continuous functions» and «derivatives», is a very simple task, which can be achieved by means of four elementary axioms. The extension to the general case can be carried out by the method of projective limits, which corresponds in this case to the Schwartz's «Principe du recollement des morceaux».

However the case of distributions of infinite order seems to be of little importance in applications. So I shall here confine myself to distributions of finite order, which I shall for convenience call merely «distributions», without specification. But it should be observed that all the following discussion could be extended without difficulty to distributions of infinite order.

The chief purpose of these lectures is to introduce the foundations of a general theory of the integral for distributions, giving a direct justification of the heuristic methods used by physicists, specially as far as convolution and Fourier transformation for distributions are concerned. Such a theory was already developed by the Polish school, in the case of integrals on a bounded interval of $\mathbf{R}$, applying the Lojasiewicz's definition of limit of a distribution at a point of

R (see [5] and [6]). However, as will be shown, the definition of the limit of a distribution $f(x)$ as x tends to infinity, given by Mikusinski and Sikorski, does not seem to be sufficiently general. So the theory that I am proposing to discuss here concerns specially the case of integrals on unbounded intervals.

A feature of this theory, which will be somewhat surprising, is its elementary form. No elements of the theory of locally convex spaces are needed here. Definitions and theorems will appear as a simple extension of advanced calculus for undergraduate students. I must confess that my own conviction is that, in many questions concerning distributions, the intervention of topological linear methods is artificial, masking the simplicity of the essential ideas. As a matter of fact, it was the topological point of view that prevented Mikusiński and Sikorski from using the definition of limit that I shall introduce here, adopting the representation of distributions as derivatives of functions (according to the idea of Bochner), instead of as linear functionals (according to Schwartz) or as limits of sequences (according to Mikusiński-Sikorski).

The first type of representation affords at the same time a natural extension of the «o» and «O» symbols to distributions, which underlie the whole theory of the integral and permit a generalization of several convergence tests.

One of the avantages of this theory of the integral for distributions is to afford a *general definition of convolution*, which has just the usual form:

$$(f*g)(x) = \int f(x-y)\,g(y)\,dy$$

employed currently by physicists without any theoretical foundation. All the nice properties of convolution, in the most general situations arising in practice, can be proved in a very simple way, applying the rules which are valid for integrals of distributions.

The Fourier transformation also can be introduced by means of the standard integral

$$\int e^{ixy} f(y)\,dy$$

and *the whole theory of Fourier transformation for tempered distributions can be developed, without assuming any previous theory of the same transformation for functions:* we have only to follow more or less the intuitive methods employed by physicists, using them now in a quite rigorous way. This approach has *at least* a didactic advantage, since the direct theory of Fourier transformation for distributions is far more simple than the classical one. In this way, the classical difficulties can be avoided and, after all, we obtain a very simple and natural introduction to more refined theories. Thus the usual order is reversed.

But at the same time some new results, which will probably be interesting in practice, can actually be obtained by this approach.

§ 1. PRELIMINARIES

1. Notation and terminology

We shall denote by $\mathbf{N}$ the set of all positive integers, by $\mathbf{N}_0$ the set of all non-negative integers, by $\mathbf{R}$ the real field and by $\mathbf{C}$ the complex field. Consider any $n \in \mathbf{N}$; given two points $a=(a_1,\ldots,a_n)$ and $b=(b_1,\ldots,b_n)$ of $\mathbf{R}^n$ we shall write $a<b$, iff $a_j<b_j$ for $j=1,\ldots,n$, and $a\leqslant b$ iff $a_j \leqslant b_j$ for $j=1,\ldots,n$ (1). By an *interval* I in $\mathbf{R}^n$ we shall understand the cartesian product of n intervals $I_1,\ldots,I_n$ in $\mathbf{R}^n$; the interval I is said to be *degenerate*, iff at least one of the intervals I_j reduces to a point. *We shall here consider only non-degenerate intervals.*

Consider the vector space $\mathcal{C}(I)$ of all complex-valued functions $f(x)= =f(x_1,\ldots,x_n)$, which are defined and continuous on the interval I in $\mathbf{R}^n$. For each system $r=(r_1,\ldots,r_n)$ of n integers $r_k \geqslant 0$, we shall put

$$D^r = D_1^{r_1} \ldots D_n^{r_n} \quad , \quad \text{where } D_k = \frac{\partial}{\partial x_k} \quad \text{for } k=1,\ldots,n \ ,$$

and we shall denote by $\mathcal{C}^r(I)$ the set of all functions f such that $D^r f$ exists and is continuous on I in the ordinary sense, *independently of the order in which the differentiations are performed.*

On the other hand, considering for each k a fixed point c_k in $\mathfrak{I}_k$, arbitrarily chosen, we shall put for each $f \in \mathcal{C}(I)$ and $k=1,\ldots,n$:

$$\mathfrak{I}_k f(x) = \int_{c_k}^{x_k} f(x_1,\ldots,\xi_k,\ldots,x_n)\, d\xi_k$$

and, more generally, $\mathfrak{I}^r = \mathfrak{I}_1^{r_1} \ldots \mathfrak{I}_n^{r_n}$. Then $\mathfrak{I}^r$ is a *right inverse* of D^r, i. e. $D^r \mathfrak{I}^r f = f$, for all $f \in \mathcal{C}(I)$.

2. Axiomatic introduction of distributions (2)

As each derivation operator D_k is not defined on the whole space $\mathcal{C}(I)$, there arises the problem of enlarging this set, in order that the operators $D_1,\ldots,D_n$ may be extended as mappings of the enlarged set into itself, according to some

(1) The expression «iff» is an abbreviation of «if and only if».

(2) As we have said in the Introduction, we call here «distributions» the distributions of finite order according to Schwartz.

natural conditions, which we are going to state precisely, under the form of axioms. The new set will be denoted by $\mathcal{C}_\infty(I)$ and its elements will be called *distributions* on I. The set $\mathcal{C}_\infty(I)$, provided with the n basic operators $D_1,\ldots,D_n$, is just defined, up to an isomorphism, by the following system of axioms:

AXIOM 1. *If $f\in\mathcal{C}(I)$, then $f\in\mathcal{C}_\infty(I)$.*

AXIOM 2. *To each $f\in\mathcal{C}_\infty(I)$ and each $k=1,\ldots,n$, there corresponds an element D_kf of $\mathcal{C}_\infty(I)$ (the derivative of f with respect to x_k), in such a way that:* (i) *if f is a function having a derivative f'_{x_k}, with respect to x_k, in the ordinary sense, and continuous on I, then D_kf coincides with f'_{x_k};* (ii) *the operators $D_1,\ldots,D_n$ are mutually interchangeable, that is: $D_jD_kf=D_kD_jf$, for all $j,k=1,\ldots,n$, and all $f\in\mathcal{C}_\infty(I)$.*

DEFINITION 1. If r is any system $(r_1,\ldots,r_n)$ of n non negative integers, then $D^r=D_1^{r_1}\ldots D_n^{r_n}$.

AXIOM 3. *For each $f\in\mathcal{C}_\infty(I)$ there exists a system r of n integers $\geqslant 0$ and a function $F\in\mathcal{C}(I)$ such that $f=D^rF$.*

AXIOM 4. *If r is a system of n integers $r_k\geqslant 0$ and $F,G\in\mathcal{C}(I)$, then we have $D^rF=D^rG$, if and only if $F-G$ is of the form $F-G=\theta_1+\ldots+\theta_n$, where θ_k is a polynomial in x_k of degree $<r_k$ whose coefficients are continuous functions on I, independent of x_k $(k=1,\ldots,n)$.*

It can be proved that this system of axioms is both *consistent* and *categorical.*

The functions θ_k considered in axiom 4 are of the form

$$\theta_k(x)=\sum_{\nu=0}^{r_k-1} x_k^\nu\cdot a_{k\nu}(x)$$

where the coefficients $a_{k\nu}$ are continuous functions on I independent of x_k. We shall denote by $\mathcal{P}_r$ the set of all functions θ of the form $\theta=\theta_1+\ldots+\theta_k$, which we shall call *pseudo-polynomials of degree $<r$*. In turn, $\mathbf{N}_0^n$ is the set of all systems of n non negative integers.

Let us consider two distributions f and g on I. Then, according to axiom 3, there exist $F,G\in\mathcal{C}(I)$ and $r,s\in\mathbf{N}_0^n$ such that

$$f=D^rF \quad , \quad g=D^sG\,.$$

Take $p\in\mathbf{N}_0^n$ such that $r\leqslant p$ and $s\leqslant p$. Remembering that $\mathcal{J}^m$ is a right inverse

of D^m and applying the condition (i) of axiom 2, we can represent f and g in the form

$$f = D^p \widetilde{F} \quad , \quad g = D^p \widetilde{G}$$

where $\widetilde{F} = \mathfrak{J}^{p-r} F$, $\widetilde{G} = \mathfrak{J}^{p-s} G$. And so we have, *according to axiom 4:*

$$f = g \quad \text{iff} \quad \mathfrak{J}^{p-r} F - \mathfrak{J}^{p-s} G \in \mathfrak{P}_p \ .$$

On the other hand we shall write *by definition:*

$$f + g = D^p (\mathfrak{J}^{p-r} F + \mathfrak{J}^{p-s})$$

$$\alpha f = D^r (\alpha F) \quad , \quad \text{for all } \alpha \in \mathbf{C} \ .$$

By these two definitions the set $\mathcal{C}_\infty(I)$ becomes a vector space (over $\mathbf{C}$) and D_k, for each $k=1,\ldots,n$, a linear mapping of the space $\mathcal{C}_\infty(I)$ into itself.

Let now J be an interval contained in I. We shall call *restriction of* f to J and denote by $\rho_J f$, the distribution $D^r(\rho_J F)$, where $\rho_J F$ is the restriction of the function F to J in the ordinary sense. It is readily seen that ρ_J is a linear mapping of $\mathcal{C}_\infty(I)$ on to $\mathcal{C}_\infty(J)$.

3. Locally summable functions and measures as distributions

For the sake of simplicity we shall confine ourselves to the case $n=1$, since the extension to the general case offers no essential difficulty.

Let f be a locally summable function on an open interval I in $\mathbf{R}$. A *primitive function* of f is any function F of the form

$$F(x) = k + \int_c^x f(\xi) d\xi \ , \qquad \text{with arbitrary } c \in I, \ k \in \mathbf{C}.$$

Then F is a continuous function such that $F'(x) = f(x)$ almost everywhere, in the ordinary sense. Denote by $\tilde{f}$ the function such that: (i) the domain of $\tilde{f}$ is the set $\tilde{I}$ of all points x of I for which $F'(x)$ exists in the ordinary sense; (ii) $\tilde{f}(x) = F'(x)$ for all $x \in \tilde{I}$. Then $\tilde{f}$ is *equivalent* to f, that is $\tilde{f}(x) = f(x)$ almost everywhere in I, and we say that $\tilde{f}$ is the *standard function* equivalent to f. From now on, when we shall speak of locally summable functions, it will be

understood that they are standard functions. The sum of two such functions f, g is defined by the formula

$$(f+g)(x) = \frac{d}{dx}\int_c^x [f(\xi)+g(\xi)]d\xi$$

and the product αf of any $\alpha \in \mathbf{C}$ by f is defined in the usual way. Then the set of all (standard) locally summable functions on I becomes a vector space over $\mathbf{C}$. We shall denote by $\dot{L}(I)$ this space; it is easily seen that:

By assigning to each function $f \in \dot{L}(I)$ the distribution $f^* = DF$, where F is any primitive function of f, there is defined a one-to-one linear mapping of $\dot{L}(I)$ into $C_\infty(I)$ such that, if f is continuous on I, then $f = f^*$.

This proposition enables us to identify every function $f \in \dot{L}(I)$ with the distribution DF, where F is a primitive a f.

Let now μ be a measure on I, i. e. a complex valued σ-additive function defined (for example) on the family of all bounded intervals J such that $\bar{J} \subset I$. A *primitive function* of μ is any function F of the form

$$F(x) = \begin{cases} k + [c,x], & \text{if } x \geqslant c \\ k -]c,x[, & \text{if } x < c \end{cases} \qquad \text{with arbitrary } c \in I,\ k \in \mathbf{C}.$$

A necessary and sufficient condition for a function F on I to be the primitive of a measure is that F be a function of (locally) bounded variation and continuous on the right at every point x of I. In particular, F may be absolutely continuous, i. e. the primitive of some locally summable function f; *then the measure μ is identified whith the function f, so that $\mu(J) = f(J) = \int_J f(\xi)d\xi$ for every bounded interval J such that $\bar{J} \subset I$.*

We shall denote by $\mathfrak{M}(I)$ the vector space of all measures on I. It is easily seen that:

By assigning to each $\mu \in \mathfrak{M}(I)$ the distribution $f = D\widetilde{F}$, where F is any primitive of μ (and $\widetilde{F}$ the standard function equivalent to F), there is defined a one-to-one linear mapping of $\mathfrak{M}(I)$ into $C_\infty(I)$, such that, if μ is a locally summable function, then $\mu = f$.

This proposition enables us to identify every measure μ on I with the distribution $D\widetilde{F}$, where $\widetilde{F}$ a standardized primitive of f.

4. Multiplication and change of variable

We shall confine ourselves to the case $n=1$.

Let I be an interval in $\mathbf{R}$. For each $p\in\mathbf{N}_0$, we denote by $\mathcal{C}_p(I)$ the space of all distributions f of the form $f=D^pF$, with $F\in\mathcal{C}(I)$. The following theorem can be proved:

For every $p\in\mathbf{N}_0$ it is possible, in one single way, to assign to each couple (f,g) where $f\in\mathcal{C}^p(I)$ and $g\in\mathcal{C}_p(I)$, a distribution $fg\in\mathcal{C}_p(I)$ independent of p and satisfying the following conditions:

(i) *if $g\in\mathcal{C}(I)$, then fg is the product of the functions f,g in the ordinary sense;*

(ii) *if $f\in\mathcal{C}^{p+1}(I)$ and $g\in\mathcal{C}_p(I)$, then $D(fg)=f.Dg+Df.g$.*

By these conditions, if $f\in\mathcal{C}^p(I)$ and $g=D^pG$, with $G\in\mathcal{C}(I)$, the distribution fg is uniquely defined by the formula

$$fg=f\cdot D^pG=\sum_{k=0}^{p}(-1)^k\binom{p}{k}D^{p-k}(f^{(k)}G) .$$

Then it is natural to call fg *the product of* f by g.

Besides, it is proved that, by this definition, $\mathcal{C}_p(I)$ *becomes a module over the complex algebra* $\mathcal{C}^p(I)$. In particular, observing that

$$\mathcal{C}_\infty(I)=\bigcup_{p=0}^{\infty}\mathcal{C}_p(I) \quad , \quad \mathcal{C}^\infty(I)=\bigcap_{p=0}^{\infty}\mathcal{C}_p(I)$$

the space $\mathcal{C}_\infty(I)$ becomes a module over $\mathcal{C}^\infty(I)$.

The preceding theorem can be extended, replacing $\mathcal{C}_p(I)$ by the space $\mathcal{M}_p(I)$ of all distributions f of the form $f=D^pF$, where $F\in\mathcal{M}(I)$, and the condition (i) by the following one:

(i') *If $g\in\mathcal{M}(I)$, then fg is the product of f by the measure g according to the usual definition.*

Let now consider another interval I^* in $\mathbf{R}$. The following theorem can be proved. *For every $p\in\mathbf{N}_0$ it is possible, in one single way, to assign to each couple (f,g), where $f\in\mathcal{C}_p(I)$ and g is a $\mathcal{C}^1$ mapping of I^* into I such that $1/g'\in\mathcal{C}^p(I^*)$, a distribution $f\circ g$, not depending on p and satisfying the following conditions:*

(i) *if $f\in\mathcal{C}(I)$, then $(f\circ g)(x)\equiv f(g(x))$.*

(ii) *if $f\in\mathcal{C}_p(I)$ and $1/g'\in\mathcal{C}^{p+1}(I^*)$, then*

$$D(f\circ g)=g'(Df\circ g)$$

By these conditions, if $1/g' \in C^p(I^*)$ *and* $f = D^p F$, *with* $F \in C(I)$, *the distribution* $f \circ g$ *is uniquely defined by*

$$f \circ g = \left(\frac{1}{g'} D\right)^p (F \circ g)$$

Moreover it can be proved that:

If $1/g' \in C^p(I^*)$, g *defines a linear mapping* $f \to f \circ g$ *of* $C_p(I)$ *into* $C_p(I^*)$.

If in addition h *is a* C^1 *mapping of an interval* I^{**} *into* I^* *such that* $1/h' \in C^p(I^{**})$, *then*

$$(f \circ g) \circ h = f \circ (g \circ h) .$$

§ 2. LIMITS AND INTEGRALS OF DISTRIBUTIONS OF ONE VARIABLE

5. Limit of a distribution as $x \to +\infty$

Let I be an open interval unbounded on the right, i. e. of the form $I =]a, +\infty[$, with $a \in \mathbf{R}$ or $a = -\infty$. The two following definitions are well known in classical analysis:

5.1. DEFINITIONS. Let f and φ be two functions on I. The function f is said to be of *order less than* φ, iff there exist a real x_0 and a function f_0 such that

$$f = \varphi f_0 \text{ for } x > x_0 \ , \ f_0(x) \to 0 \text{ as } x \to +\infty .$$

On the other hand, f is said to be *at most of the order of* φ, as $x \to +\infty$, iff there exist a real x_0 and a function f_0 bounded for $x > x_0$ such that $f = \varphi f_0$.

In the first case we shall write

$$f \in o(\varphi) \text{ as } x \to +\infty \qquad (\text{or } \textit{on the right})$$

and in the second case

$$f \in O(\varphi) \text{ as } x \to +\infty \qquad (\text{or } \textit{on the right}) .$$

These notations replace the classical ones $f = o(\varphi)$ and $f = O(\varphi)$, which are not logically correct and may produce some confusion in functional analysis.

Observe that

5.2 *If there exists* x_0 *such that* $\varphi(x)\neq 0$ *for* $x>x_0$, *then*

$$f\in o(\varphi) \quad \text{as} \quad x\to+\infty \iff \frac{f(x)}{\varphi(x)}\to 0 \quad \textit{as} \quad x\to+\infty$$

$$f\in O(\varphi) \quad \text{as} \quad x\to+\infty \iff \frac{f(x)}{\varphi(x)} \textit{ is bounded on the right} .$$

In order to extend the symbol O to distributions, we shall consider at first the case when $\varphi=\hat{x}^\alpha$ with $\alpha>-1$ (for simplicity the sign ^, *of dummy variable*, will be omitted). Let $\mathcal{J}$ be the integration operator defined by $\mathcal{J}f(x)=\int_c^x f(\xi)d\xi$ with c in I.

5.3. Lemma. *If* α *is a real number* >-1 *and* f *a continuous function such that* $f\in o(x^\alpha)$ *as* $x\to+\infty$, *then*

$$\mathcal{J}f\in o(x^{\alpha+1}) \quad \textit{as} \quad x\to+\infty .$$

Proof. Suppose $f\in o(x^\alpha)$ as $x\to+\infty$. This means that there exist x_0 and f_0 such that $f=x^\alpha f_0$ for $x>x_0$ and $f_0\to 0$ as $x\to+\infty$. Let ε be an arbitrary positive number; then there exists x_1 such that $|f_0(x)|<\varepsilon$ for $x>x_1$. We may assume of course $x_1>x_0>0$. Now, for every $x\in I$:

$$\mathcal{J}f(x)=k+\int_{x_1}^x \xi^\alpha f_0(\xi)d\xi \quad , \quad \text{where } k=\int_c^{x_1} f(\xi)d\xi$$

Since $|f_0(\xi)|<\varepsilon$ and $\xi>0$ for $\xi>x_1$ we have

$$\left|\frac{\mathcal{J}f(x)}{x^{\alpha+1}}\right|\leqslant\frac{|k|}{x^{\alpha+1}}+\frac{(x-x_1)^{\alpha+1}}{(\alpha+1)x^{\alpha+1}}\,\varepsilon \quad , \quad \text{for } x>x_1 ,$$

and therefore, remembering that $\alpha>-1$:

$$\overline{\lim_{x\to+\infty}}\left|\frac{\mathcal{J}f(x)}{x^{\alpha+1}}\right|\leqslant\frac{\varepsilon}{\alpha+1} .$$

As ε is arbitrary, this implies that $\mathcal{J}f(x)/x^{\alpha+1} \to 0$ as $x \to +\infty$, i. e. $\mathcal{J}f \in o(x^{\alpha+1})$ as $x \to +\infty$.

5.4. Remark. This lemma extends obviously to locally summable functions and even to measures.

The lemma suggests the following:

5.5. Definition. Let α be a real number > -1 and f a distribution on I. We write: $f \in o(x^{\alpha})$ as $x \to +\infty$, iff there exist an integer $p \geqslant 0$ and a continuous function F on I, such that

$$f = D^p F \quad \text{and} \quad \frac{F(x)}{x^{\alpha+p}} \to 0 \quad \text{as } x \to +\infty .$$

5.6. Remark. The lemma implies that, *if there exist* $p \in \mathbf{N}_0$ *and* $F \in \mathcal{C}(I)$ *satisfying the preceding conditions, then every integer* $m \geqslant p$ *and every function* G *such that* $G = \mathcal{J}^{m-p} F + P$, *where* $P \in \mathcal{P}_m$, *satisfy the same conditions* (observe that if $P \in \mathcal{P}_m$, then $P(x)/x^{\alpha+m} \to 0$ as $x \to +\infty$).

5.7. Linearity Property. *If* $f \in o(x^{\alpha})$ *and* $g \in o(x^{\alpha})$ *as* $x \to +\infty$, *with* $\alpha > -1$, *then*

$$\lambda f + \mu g \in o(x^{\alpha}) \quad \textit{as} \quad x \to +\infty \quad \textit{for all} \quad \lambda, \mu \in \mathbf{C} .$$

For the proof, it is sufficient to represent f, g as derivatives of the same order of continuous functions, taking 5.6 into account.

In particular, α may be equal to 0. Then $x^0 = 1$ and, if $f \in o(1)$ as $x \to +\infty$, it is natural to say that $f \to 0$ as $x \to +\infty$. More generally, let λ be any complex number and $f \in \mathcal{C}_{\infty}(I)$; then:

5.8 Definition. We say that f *converges* to λ as $x \to +\infty$, iff $f - \lambda \in o(1)$ as $x \to +\infty$. A distribution f is said to be *convergent* as $x \to +\infty$, iff there exists $\lambda \in \mathbf{C}$ such that $f \to \lambda$ as $x \to +\infty$.

Taking the definition 5.5 into account and observing that $\lambda = D^p(\lambda^p/p!)$ for every $p \in \mathbf{N}_0$, we can define directly the preceding concept as follows:

5.9. Definition. We say that $f \to \lambda$ as $x \to +\infty$, iff there exist $p \in \mathbf{N}_0$ and $F \in \mathcal{C}(I)$ such that

$$f = D^p F \quad \text{and} \quad \frac{F(x)}{x^p} \to \frac{\lambda}{p!} \quad \text{as } x \to +\infty \text{ (in the ordinary sense).}$$

REMARK. Instead of «f tends to λ as $x\to+\infty$», we shall write sometimes «$f(x)\to\lambda$ as $x\to+\infty$». But it should be remembered that in these cases «x» is a dummy variable.

5.10. *If* $f\to\lambda$ *as* $x\to+\infty$ *and* $f\to\mu$ *as* $x\to+\infty$, *then* $\lambda=\mu$.

In fact, if $f-\lambda\to 0$ and $f-\mu\to 0$ as $x\to+\infty$, then, by 5.7, $(F-\lambda)-(F-\mu)=$ $=\mu-\lambda\to 0$ as $x\to+\infty$. But, for every integer $p\geqslant 0$ and every continuous function F such that $\mu-\lambda=D^pF$, we have necessarily $F=(\mu-\lambda)x^p/p!+P$, where P is a polynomial of degree $<p$. Hence, according to def. 5.9, $\mu-\lambda$ cannot tend to 0, *unless* $\mu=\lambda$.

This legitimates the definition complementary to 5.8:

5.11. DEFINITION. We say that λ is *the limit of f as $x\to+\infty$*, iff $f\to\lambda$ as $x\to+\infty$. In this case, we shall write $\lambda=\lim\limits_{x\to+\infty} f(x)$ or $\lambda=f(+\infty)$.

The uniqueness of the limit is guaranteed just by 5.10. Besides, from 5.7 follows:

5.12. LINEARITY PROPERTY. *If f and g are convergent as $x\to+\infty$, then*

$$\lim_{x\to+\infty}(\alpha f+\beta g)=\alpha\lim_{x\to+\infty} f+\beta\lim_{x\to+\infty} g \quad , \quad \forall\alpha,\beta\in\mathbf{C}$$

In turn, from 5.3 and the preceding definitions, it follows:

5.13. *If f is a continuous function such that $\lim\limits_{x\to+\infty} f(x)=\lambda$ in the ordinary sense, then the same fact holds in the distributional sense, i. e. in the sense of definitions* 5.4 *and* 5.8.

Observe that, according to 5.4, this theorem extends to locally summable functions (and even to measures). However, it must be observed that *the converse of this theorem is not true:*

5.14. EXAMPLE. As is well known, the function $\cos x$ is not convergent, in the *ordinary sense*, as $x\to+\infty$. But we have

$$\lim_{x\to+\infty}\cos x=0\ , \ \textit{in the distributional sense}$$

To see that, it is enough to apply def. 5.4, observing that $\cos x=D\sin x$ and $\dfrac{\sin x}{x}\to 0$ as $x\to+\infty$.

5.15. GENERAL REMARK. *All preceding definitions may be extended and all propositions remain true, if we replace throughout $+\infty$ by $-\infty$ and «on the right»*

by «on the left». In particular, we must then consider an interval I unbounded on the left, $I=]-\infty, a[$, instead of an interval unbounded on the right.

5.16. Definition. We say that f tends to λ as $x \to \infty$ and we write $\lim\limits_{x\to\infty} f(x)=\lambda$, iff $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \lambda$.

For example, it is easily seen that (cf. 5.14):

$$\lim_{x\to\infty} \sin x = 0 \quad \textit{(in the distributional sense)} .$$

6. Limit and value of a distribution at a point of $\mathbf{R}$

Let now I be any open interval $]a,b[$, bounded on the left, i. e. with $a\in\mathbf{R}$ (but $b\in\mathbf{R}$ or $b=+\infty$). Then, definitions 5.1 are readily extended to this case, replacing throughout «$x\to+\infty$» by «$x\to a^+$» and «on the right» by «on the left». Besides, if we put $\mathcal{J}f(x) = \int_a^x f(\xi)d\xi$, we prove, as we did for 6.1.3 (the proof is even simpler):

6.1. Lemma. *If f is a continuous function on I, such that $f\in O((x-a)^\beta)$ as $x\to a^+$, where β is a real number >-1, then*

$$\mathcal{J}^n f\in O\big((x-a)^{\beta+n}\big) \quad \textit{as } x\to a^+ \quad , \quad \textit{for } n=0,1,\ldots$$

This lemma justifies the following:

6.2. Definition. If $f\in C_\infty(I)$ and $\beta>-1$, we write $f\in o(x^\beta)$ as $x\to a^+$, iff there exist $p\in\mathbf{N}_0$ and $F\in C(I)$ such that $f=D^pF$ and $F(x)/(x-a)^{\beta+p}\to 0$ as $x\to a^+$.

6.3 Remark. *The lemma implies that, if there exist p and F satisfying these conditions, then every integer $m\geqslant p$, along with the function $\mathcal{J}^{m-p}F$, satisfies the same conditions* (but it must be observed that for each integer $m\geqslant p$, there is no function different from $\mathcal{J}^{m-p}F$ satisfying the same conditions).

Now we are able to extend definitions 5.8 and 5.11, as well as propositions 5.7, 5.10, 5.12 and 5.13, replacing $+\infty$ by a^+. In particular, the convergence as $x\to a^+$ can be defined directly as follows:

6.4. Definition. A distribution f on $I=]a,b[$ tends to λ as $x\to a^+$, iff there exist $p\in\mathbf{N}_0$ and $F\in C(I)$, such that

$$f=D^pF \quad \text{and} \quad \frac{F(x)}{(x-a)^p}\to\frac{\lambda}{p!} \text{ as } x\to a^+ \text{ (in the ordinary sense)}.$$

Besides, the concepts of convergence corresponding to the cases $x\to+\infty$ and $x\to a^+$ are related to each other according to the following rule:

6.5. *Suppose $I=]a,+\infty[$ and $f\in C_\infty(I)$. Then, if $g(t)=f(a+1/t)$, we have:*

$$\lim_{t\to+\infty} g(t)=\lambda \iff \lim_{x\to a^+} f(x)\,.$$

Proof. We can obviously reduce to the case $a=0$ and $\lambda=0$. Suppose $f(x)\to 0$ as $x\to 0^+$. Then, there exist $p\in\mathbf{N}_0$ and $F\in C(I)$ such that $f=D^pF$ and $F(x)/x^p\to 0$ as $x\to a^+$. Moreover (cf. 4) we have $g(t)=(-t^2D_t)^pF(1/t)$ and it is easily shown by induction on p that there exist $p+1$ numbers a_k (whose expressions are not needed here) such that

$$\text{6.6.} \qquad g(t)=\sum_{k=0}^{p} a_k D_t^k\left[t^{p+k}F\left(\frac{1}{t}\right)\right]$$

Now, since $F(x)/x^p$ tends to 0 as $x\to 0^+, t^pF(1/t)$ tends to 0 as $t\to+\infty$. Hence

$$\lim_{t\to+\infty}\frac{t^{p+k}F(1/t)}{t^k}=0\,, \qquad \text{for } k=0,\dots,p\,,$$

which according to def. 5.9, means that all terms in the right side of 6.6 tend to 0 as $t\to+\infty$. Therefore, according to the linearity property, $g(t)$ tends to 0 as $t\to\infty$. In a similar way, we prove that, if $g(t)\to 0$ as $t\to+\infty$ then $f(x)\to 0$ as $x\to 0^+$. We can obviously define the meaning of

$$f(x)\to\lambda \qquad \text{as } x\to b^-\,,$$

as we did for the case $x\to a^+$ considering now an interval $]a,b[$ *bounded on the right.* It is readily seen that all preceding propositions and remarks can be extended to this case.

Let I be now any open interval in $\mathbf{R}$, $I=]a,b[$, and let c be any point of I, that is $a<c<b$. Then, if $f\in\mathcal{C}_\infty(I)$, we define the meanings of

$$«f(x)\to\lambda \text{ as } x\to c^+» \quad \text{and} \quad «f(x)\to\lambda \text{ as } x\to c^-»$$

by considering, instead of f, *its restrictions* to the intervals $]a,c[$ and $]c,b[$.

As in classical analysis, we shall put

$$f(c^+)=\lim_{x\to a^+} f(x) \qquad \textit{(right-hand limit of } f \textit{ at } c\textit{)}$$

$$f(c^-)=\lim_{x\to a^-} f(x) \qquad \textit{(left-hand limit of } f \textit{ at } c\textit{)}$$

whenever the limit in question exists.

6.7. Definition. We say that *f tends to λ as $x\to c$*, iff $f(x)\to\lambda$ as $x\to c^+$ *and* $f(x)\to\lambda$ as $x\to c^-$. In this case we write $\lambda=\lim\limits_{x\to c} f(x)$.

According to the preceding definitions and remarks, we can also define directly this concept:

6.8. Definition. The distribution *f tends to λ as $x\to c$*, iff there exist an integer $p\geqslant 0$ and a function F continuous at every point x of I *distinct from c*, such that

$$f=D^pF \quad \text{and} \quad \lim_{x\to c}\frac{F(x)}{(x-c)^p}=\frac{\lambda}{p!} \quad \textit{in the ordinary sense}\,.$$

Remark. Suppose that I is, more generally, *any* (non-degenerate) interval I in $\mathbf{R}$ and c is an inner point or an extremity of I. Then definition 6.8 applies, *even* if is c an extremity of the domain I of f; for example, if c is the left extremity of I, we have by definition $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c^+} f(x)$.

With the general hypothesis considered above, we have:

6.9. Definition. A distribution f on I is said to be *continuous at the point c*, iff there exist $p\in\mathbf{N}_0$ and $F\in\mathcal{C}(I)$ such that $f=D^pF$ and $F(x)/(x-c)^p$ is convergent in ordinary sense as $x\to c$. Then we write

$$f(c)=\lim_{x\to c} f(x)=p!\lim_{x\to c}\frac{F(x)}{(x-c)^p}$$

and the number $f(c)$ is said to be the *value* of the distribution f at the point c (or for $x=c$).

Examples — I. Consider $f(x)=\cos 1/x$. Then f is a locally summable function on $\mathbf{R}$ and, since

$$\cos \frac{1}{x} = 2x \sin \frac{1}{x} - D\left(x^2 \sin \frac{1}{x}\right), \quad \lim_{x\to 0}\left(x \,.\, \sin \frac{1}{x}\right) = 0$$

it is easily seen that f is continuous at the point 0 with the value 0 (in the distributional sense, not in the ordinary sense!). Observe that the right member has the value 0 *even in the ordinary* sense after the functions have been standardized.

II. It can be seen, as an exercise, that $\lim\limits_{x\to 0} \delta^{(k)}=0$ and yet $\delta^{(k)}$ is not continuous at 0 for any $k=0,1,\ldots$

III. It can be proved, as an exercise, that: *If f is a distribution on an interval I minus a point c of $\mathbf{R}$ belonging to $\bar{I}$, and if f is convergent as $x\to c$, then there exists one, and only one distribution $\tilde{f}$ on I plus c, which is continuous at c and such that $\tilde{f}=f$ on I.*

Remark. The previous concepts of limit and value of a distribution at a point of $\mathbf{R}$ have been introduced by Lojasiewicz [5]. As for the concepts of limit as $x\to+\infty$ or as $x\to-\infty$, the definitions given by Mikusiński and Sikorski [6] seem to be too restrictive, as they are not invariant for very simple substitutions, such as $x=1/t$ and do not allow to justify certain integral formulas occuring in applications. The definitions that we are using here do not present these disadvantages.

7. Primitives and integrals of distributions

If f is a distribution with domain in $\mathbf{R}$, we call *primitive of f* any distribution φ such that $D\varphi=f$. According to this definition:

7.1. Theorem. *Every distribution f on an interval I has infinitely many primitives and any two primitives of f differ by a constant.*

Let f be a distribution on I. Then f is of the form $f=D^nF$, with $F\in\mathcal{C}(I)$, and every distribution φ of the form $\varphi=D^n\mathfrak{J}F+k$, where $\mathfrak{J}$ is an integration operator and $k\in\mathrm{C}$, is obviously a primitive of f. Suppose now $D\varphi_1=D\varphi_2=F$; then, if $\varphi_1=D^n\Phi_1$ and $\varphi_2=D^n\Phi_2$, with $\Phi_1,\Phi_2\in\mathcal{C}(I)$, we have $D^{n+1}\Phi_1=D^{n+1}\Phi_2$, which implies, according to axiom 4 (cf. 2), that $\Phi_1-\Phi_2$ is a polynomial P of degree $<n+1$. Thus $\varphi_1-\varphi_2=D^nP=const.$

From here and 6.9 it follows immediately:

7.2. Corollary. *If there exists a primitive of f which is continuous at a point a, then every primitive of f is continuous at a, and, for every complex number k, there exists one, and only one, primitive φ of f such that $\varphi(a)=k$.*

It will be natural to denote by the symbol.

$$\int_a^{\hat{x}} f(\xi)d\xi \qquad \text{or shortly by} \qquad \int_a^{\hat{x}} f$$

the primitive of f assuming the value 0 at a. (Remember that the sign $\hat{}$, indicating that x is a dummy variable, may be omitted, whenever no confusion is possible). Thus, according to 7.2, if there exists at least one primitive of f which is continuous at the point a, the *differential equation* $D\varphi=f$ will have one single solution satisfying the *initial condition* $\varphi(a)=k$; and such a solution is

$$\varphi(x)=k+\int_a^x f(\xi)\,d\xi\ .$$

As we observed, it is understood that here x is only a dummy variable: the distribution φ need not actually have any value $\varphi(x)$ at *every* point x of I. But, obviously, if φ has a value at *some* point b of I, this value will be given by the formula

$$\varphi(b)=k+\int_a^b f(\xi)d\xi\ .$$

Thus the integral $\int_a^b f(\xi)d\xi$ (in short $\int_a^b f$) is *defined* by the generalized Barrow's Formula:

$$\int_a^b f(x)dx=\varphi(b)-\varphi(a)\ .$$

The corollary 7.2 can be extended as follows:

7.3. Corollary. *If there exists a primitive of f having a limit as $x\to a^+$ [resp. as $x\to a^-$], then, for every complex number k, there exists one, and only one, primitive φ of f such that* $\varphi(a^+)=k$ [resp. $\varphi(a^-)=k$].

Remember that the existence of both $\varphi(a^+)$ and $\varphi(a^-)$ does not imply the existence of $\varphi(a)$.

All preceding remarks and conventions may now be extended analogously to new cases. For example, we shall denote by $\int_{a^-}^{x} f(\xi)d\xi$ (in short $\int_{a^-}^{\hat{x}} f$) the primitive of f on I which tends to zero as $x \to a^-$; accordingly, if such a limit exists, the differential equation $D\varphi = f$ along with the initial condition $\varphi(a^-) = k$ will have the only solution·

$$\varphi(x) = k + \int_{a^-}^{x} f(\xi)d\xi \ .$$

So we shall have by definition:

$$\int_{a^-}^{b^-} f(x)dx = \varphi(b^-) - \varphi(a^-) \ , \quad \int_{a^-}^{b^+} f(x)dx = \varphi(b^+) - \varphi(a^-) \ .$$

If $a < b$, these are, respectively, *the integrals of the distribution on the intervals* $[a, b[$ *and* $[a, b]$. The integrals of f on $]a, b]$ and $]a, b[$ are analogously defined. Naturally, such an integral is said to *exist* or to be *convergent* iff the two corresponding limits exist. If $b \leqslant a$, we have of course

$$\int_{a^-}^{b^+} f = -\int_{b^+}^{a^-} f \quad , \quad \int_{a^+}^{b^+} f = -\int_{b^+}^{a^+} f \quad , \quad \text{etc.}$$

Finally all preceding definitions may be extended to *infinite intervals.* For example, we have by definition:

$$\int_{a^-}^{+\infty} f(x)dx = \varphi(+\infty) - \varphi(a^-) \ ,$$

if φ is a primitive of f such that the limits on the right side exist; and $\int_{a^-}^{+\infty} f(x)\,dx$ is called the integral of f *on the interval* $[a, +\infty[$. For other kinds of infinite intervals such as $]-\infty, a[$, $]-\infty, +\infty[$, etc. the definitions are quite analogous.

In general case, a distribution f is said to be *integrable on an interval* I, iff the integral of f on I exists. This integral may be denoted by $\int_I f(x)dx$ or simply by $\int_I f$.

From the linearity property of limits it follows immediately the corresponding property for integrals:

7.4. Linearity Property. *If two distributions f and g are integrable on I, so is $\alpha f+\beta g$, for any $\alpha,\beta\in\mathbf{C}$ and*

$$\int_I (\alpha f+\beta g)=\alpha\int_I f+\beta\int_I g\ .$$

On the other hand, it should be observed that:

7.5. *If f is a function summable on I, then the integral of f on I, in the distributional sense, exists and equals the Lebesgue integral on I. More generally, if f is a locally summable function on I, such that $\int_I f$ is convergent in the classical sense (even simply convergent), then $\int_I f$ exists with the same value, in distributional sense.*

However, the converse of this proposition is not true, as we shall presently see. Consider the integral $\int_{\mathbf{R}} e^{i\omega t}dt$, where ω is a real parameter. This integral is obviously divergent, in classical sense, for every value of ω. However, for $\omega\neq 0$, one primitive of $e^{i\omega t}$ is $e^{i\omega t}/i\omega$ and

$$\frac{e^{i\omega t}}{i\omega}=\frac{1}{(i\omega)^2}D_t e^{i\omega t}\quad,\quad \lim_{t\to\infty}\frac{e^{i\omega t}}{t}=0\ .$$

Hence, we have in the distributional sense, for every $\omega\neq 0$:

$$\int_{-\infty}^{+\infty} e^{i\omega t}dt=\frac{1}{i\omega}\left(\lim_{t\to+\infty}e^{i\omega t}-\lim_{t\to-\infty}e^{i\omega t}\right)=0\ .$$

For $\omega=0$ this integral is divergent, even in the distributional sense. These results agree with the intuition of physicists, which have, since long, adopted the formula

7.6. $$\int_{\mathbf{R}} e^{i\omega t}dt=2\pi\delta(\omega)\ .$$

However, a complete justification of this formula cannot be achieved, without a suitable definition of *parametric integral*, which will be given in § 3. Let us now prove the following proposition:

7.7. *Every distribution with a bounded carrier on* $\mathbf{R}$ *is integrable on* $\mathbf{R}$.

Proof. Let f be a distribution of bounded carrier on $\mathbf{R}$. This means that there exists a bounded interval $I=[a,b]$ such that $f=0$ outside I. Hence, if φ is a primitive of f, $D\varphi=0$ outside I and φ reduces to constants c_1 and c_2, respectively on $]-\infty,a[$ and on $]b,+\infty[$. Thus

$$\varphi(-\infty)=\varphi(a^-)=c_1 \qquad \text{and} \quad \varphi(b^+)=\varphi(+\infty)=c_2\,.$$

Hence f is integrable on $\mathbf{R}$ and $\int\limits_{\mathbf{R}} f=\int\limits_{I} f=\int\limits_{a^-}^{b^+} f=c_2-c_1$.

8. Orders of growth for distributions

For brevity, we shall confine ourselves to the typical case when $x\to+\infty$, since the considerations in the other cases are analogous. Let I be any interval unbounded on the right and $\mathcal{J}f(x)=\int\limits_{c}^{x} f$ with $c\in I$, for $f\in\mathcal{C}(I)$. The extension of the «O» symbol to distributions is based on the following lemma, whose proof is similar to the one of 5.3, and even more simple:

8.1. LEMMA. *If f is a continuous function on I such that $f\in O(x^\alpha)$ as $x\to+\infty$ with $\alpha>-1$, then $\mathcal{J}f\in O(x^{\alpha+1})$ as $x\to+\infty$.*

8.2. DEFINITION. If $f\in\mathcal{C}_\infty(I)$ and $\alpha>-1$, then we write $f\in O(x^\alpha)$ as $x\to+\infty$, iff there exist $x\in\mathbf{N}_0$ and $F\in\mathcal{C}(I)$ such that: $f=D^nF$ and $F(x)/x^{\alpha+n}$ is bounded on the right.

The lemma guarantees the linearity property for this case. In particular:

8.3. DEFINITION. A distribution f on I is said to be *bounded on the right*, iff $f\in O(1)$ as $x\to+\infty$, that is, iff there exist $n\in\mathbf{N}_0$ and $F\in\mathcal{C}(I)$ such that $f=D^nF$ and $F(x)/x^n$ is bounded on the the right.

Now we are able to define the meaning of expressions such that «$f\in o(\varphi)$» and «$f\in O(\varphi)$» in the more general case when $f\in\mathcal{C}_\infty(I)$ and $\varphi\in C^\infty(I)$. For that purpose, we can take as a model the classical definitions 5.1:

8.4. DEFINITIONS. We shall write $f\in o(\varphi)$ as $x\to+\infty$, iff there exist a real x_0 and a distribution f_0 such that: $f=\varphi f_0$ for $x>x_0$ and $f_0\to0$ as $x\to+\infty$.

We shall write $f \in \mathrm{O}(\varphi)$ as $x \to +\infty$, iff there exist a real x_0 and a distribution f_0 such that: $f = \varphi f_0$ for $x > x_0$ and f_0 is bounded on the right.

The first thing to do is to see whether these definitions are equivalent to the previous ones, in the particular case when φ is of the form x^α, with $\alpha > -1$. This is easily proved, by means of the formulas

$$x^\alpha . D^n F_0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} D^{n-k}(F_0 \,.\, D^k x^\alpha)$$

$$D^n(x^\alpha G_0) = \sum_{k=0}^{n} \binom{n}{k} (D_x^k x^\alpha)(D^{n-k} G_0)$$

taking into account the linearity property.

On the other hand, this same property can be now immediately extended to the general case. Moreover definitions 8.4 introduce a remarkable new property, which is a counterpart of the preceding lemmas:

8.5. Differenciation Property. *If $f \in \mathrm{O}(x^\alpha)$ on the right, then $Df \in \mathrm{O}(x^{\alpha-1})$ on the right, for every $\alpha \in \mathbf{R}$.*

We shall begin the proof in the case $\alpha = 0$:

8.6. *If f is bounded on the right, then $Df \in \mathrm{O}(x^{-1})$ as $x \to +\infty$.*

Suppose f bounded on the right. Then there exist $p \in \mathbf{N}_0$, $F \in \mathcal{C}(I)$ and c such that $f = D^p F$ for $x > c$ and $F(x)/x^p$ is bounded on the right. We may choose $c > 0$; then we have $Df = x^{-1}(x \,.\, D^{p+1} F) = x^{-1}[D^{p+1}(xF) - D^p F]$ for $x > c$ and it is readily seen that $D^{p+1}(xF)$ is bounded on the right, as well as $D^p F$. Hence $Df \in \mathrm{O}(x^{-1})$ as $x \to +\infty$.

Suppose now $f \in \mathrm{O}(x^\alpha)$ as $x \to +\infty$, where $\alpha \in \mathbf{R}$. Then there exist x_0 and f_0 such that $f = x^\alpha f_0$ for $x > x_0$ and $f_0 \in \mathrm{O}(1)$ on the right. It follows that $Df = \alpha x^{\alpha-1} . f_0 + x^\alpha . Df_0$, and it is readily seen, applying 8.6, that $Df \in \mathrm{O}(x^{\alpha-1})$ as $x \to +\infty$.

By an identical argument it is shown that the *differenciation property extends to the «o» symbol.*

Furthermore it is a simple matter to prove the following properties, where the expression «on the right» or «as $x \to +\infty$» is omitted for simplicity:

8.7. *If f is convergent, then f is bounded.*

8.8. *If $f \in \mathrm{o}(\varphi)$ then $f \in \mathrm{O}(\varphi)$.*

8.9. *If* $f \in O(x^{\alpha})$ *and* $\alpha < \beta$, *then* $f \in o(x^{\beta})$.

Obviously, we have chosen the case when $x \to +\infty$ as a model; concepts and properties are quite analogous in cases when $x \to -\infty$, $x \to c^{+}$, etc.

8.10. Convention. If a distribution f has the same growth property as $x \to +\infty$ and as $x \to -\infty$, we shall say simply that f has this property as $x \to \infty$. If a a distribution f on an interval I is bounded both on the right and on the left (i. e. as x tends to the right extremity and to the left extremity), we shall say that f is *bounded on* I or simply *bounded.*

Remark. The concept of bounded distribution that we have just introduced is more general than the concept of bounded distribution according to Schwartz, and it is necessary for the integral theory, as we shall next see.

9. Convergence tests for integrals

Let us consider, at first, the case of integrals on $\mathbf{R}$. We have then the following test, which is not true in classical analysis:

9.1. (A Necessary Condition For Convergence). *If a distribution* f *is integrable on* $\mathbf{R}$, *then* $f \in O(x^{-1})$ *as* $x \to \infty$.

Proof. Suppose that there exists a primitive φ of f such that φ is convergent as $x \to +\infty$ and as $x \to -\infty$ [1]. Then, by 8.7, φ is bounded on $\mathbf{R}$ and, by 8.5 (and its analogue for the case $x \to -\infty$), we have $\varphi \in O(x^{-1})$ as $x \to \infty$.

On the other hand, the following theorem extends to distributions a well known classical test:

9.2 (A Sufficient Condition For Convergence). *If there exists a number* $\alpha < -1$ *such that* $f \in O(x^{\alpha})$ *as* $x \to \infty$, *then* f *is integrable on* $\mathbf{R}$.

Proof. Suppose $f \in O(x^{\alpha})$ as $x \to \infty$, with $\alpha < -1$. Then there exist a number $c > 0$, an integer $n \geqslant 0$ and a continuous function F, such that

$$f = x^{\alpha} D^{n} F \text{ for } |x| > c \quad , \quad \text{with } \frac{F(x)}{x^{n}} \text{ bounded for } |x| > c$$

---

[1] This does not mean that f is convergent as $x \to \infty$, for the limits are in general different.

Set

$$F_1(x) = \begin{cases} F(x) \ , & \text{for } |x| > c \\ 0 \ , & \text{for } |x| < c \end{cases} \ , \quad f_1 = x^\alpha D^n F_1 \ , \quad f_2 = f - f_1 \, .$$

Then f_2 is a distribution with support contained in $[-c, c]$, hence integrable on $\mathbf{R}$. So we have only to prove that f_1 is integrable on $\mathbf{R}$, for then we have $\int_{\mathbf{R}} f = \int_{\mathbf{R}} f_1 + \int_{\mathbf{R}} f_2$. We shall put for simplicity $f_1 = f$ and $F_1 = F$. Then

$$f = x^\alpha D^n F = \sum_{k=0}^{n} (-1)^k c_k D^{n-k}(x^{\alpha-k} F)$$

where $c_k = \alpha(\alpha-1)\ldots(\alpha-k+1)\binom{n}{k}$. From here we deduce the following primitive of f:

9.3. $$\varphi = \sum_{k=0}^{n-1} (-1)^k c_k D^{n-k-1}(x^{\alpha-k} F) + (-1)^n c_n \int_0^x \xi^{\alpha-n} F(\xi) d\xi \, .$$

But, since $F \in \mathrm{O}(x^n)$ as $x \to \infty$, in the ordinary sense, we have $\xi^{\alpha-n} F \in \mathrm{O}(\xi^\alpha)$ as $x \to \infty$, with $\alpha < -1$, and, according to the classical test, this implies that the function $\xi^{\alpha-n} F$ is summable on $\mathbf{R}$. Hence the last term in 9.3 is convergent as $x \to +\infty$ and as $x \to -\infty$.

As to the other terms, observe that the functions

$$\frac{x^{\alpha-k} F(x)}{x^{n-k-1}} = x^{\alpha+1} \frac{F(x)}{x^n} \qquad \text{for } k = 0, \ldots, n-1$$

tend to 0 as $x \to \infty$, since $\alpha+1 < 0$ and $F(x)/x^n$ is bounded (in the ordinary sense). Hence, by def. 5.9,

$$D^{n-k-1}(x^{\alpha-k} F) \to 0 \qquad \text{as } x \to \infty$$

so that

$$\varphi(+\infty) = (-1)^n c_n \int_0^{+\infty} x^{\alpha-n} F(x) dx \quad , \quad \varphi(-\infty) = (-1)^n c_n \int_0^{-\infty} x^{\alpha-n} F(x) dx \, .$$

Therefore f is integrable on $\mathbf{R}$ and

$$\int_{\mathbf{R}} f = \int_{\mathbf{R}} x^{\alpha} D^n F = (-1)^n \int_{\mathbf{R}} F(x) \frac{d^n x^{\alpha}}{dx^n}\, dx\,.$$

We can deduce similar tests for integrals on intervals distinct from $\mathbf{R}$. For example, consider an interval $I =]a, +\infty[$ and $f \in C_{\infty}(I)$. Then it is easily seen that:

If f is integrable on I, then $f \in O(x^{-1})$ as $x \to +\infty$ and $f \in O\big((x-a)^{-1}\big)$ as $x \to a^+$.

If there exist $\alpha < -1$ and $\beta > -1$ such that $f \in O(x^{\alpha})$ as $x \to +\infty$ and $f \in O\big((x-a)^{\beta}\big)$ as $x \to a^+$, then f is integrable on I.

10. Multiplication and change of variable in connection with limits and integrals

It is a simple matter to prove the following propositions:

10.1. *If a distribution f on an interval I is convergent as x tends to c^+ with $c \in I$, and if $g \in C^{\infty}(I)$, then fg is convergent as $x \to c^+$ and*

$$\lim_{x \to c^+} (fg) = (\lim_{x \to c^+} f)(\lim_{x \to c^+} g)\,.$$

10.2. *If a distribution f on an interval I is convergent as $x \to c^+$ with $c \in I$, and if φ is an increasing C^{∞} mapping of an interval I^* into I, then $f(\varphi(t))$ is convergent as $t \to \gamma^+$, with $\varphi(\gamma) = c$, and*

$$\lim_{t \to \gamma^+} f(\varphi(\tau)) = \lim_{x \to c^+} f(x)\,.$$

Obviously, these two propositions can be extended to the case when f is convergent as $x \to c^-$. Then the second one enables the usual substitution property to be extended to integrals of distributions on bounded intervals. For example, assuming that $f \in C_{\infty}(I)$, $a, b \in I$ and φ is an increasing C^{∞} mapping of I^* into I such that $a = \varphi(\alpha)$, $b = \varphi(\beta)$, we have

$$\int_{a^+}^{b^-} f(x)dx = \int_{\alpha^+}^{\beta^-} f(\varphi(t))\varphi'(t)dt$$

whenever the first integral exist.

However these criteria are not sufficient in certain cases which occur in practice. Our next purpose is to introduce a stronger criterion than 10.2. For simplicity, we shall reduce our discussion to the case when $x \to +\infty$ and $\varphi(+\infty) = +\infty$, which can be taken as a model for other cases:

10.3. THEOREM. *Let f be a distribution on an interval I unbounded on the right and φ a C^∞ mapping of an interval I^* into I such that $\varphi'(t) \neq 0$ on I^* and $\varphi(t) \to +\infty$ as $t \to +\infty$. Suppose that:*

(i) *f is convergent as $x \to +\infty$.*

(ii) *φ' tends to a number $c \neq 0$ as $t \to +\infty$ (in the ordinary sense).*

(iii) *$\varphi^{(k+1)} \in o(t^{-k})$ as $t \to +\infty$ (in the ordinary sense) for all $k = 1, 2, \ldots$*

Then we have

$$\lim_{t \to +\infty} f(\varphi(t)) = \lim_{x \to +\infty} f(x)$$

Proof. Suppose $f \to \lambda$ as $x \to +\infty$. Then there exist $n \in \mathbf{N}_0$ and $F \in C(I)$ such that $f = D^n F$ and $F(x)/x^n \to \lambda/n!$ as $x \to +\infty$. Now $f \circ \varphi = [(1/\varphi')D_t]^n (F \circ \varphi)$ and, according to the hypothesis,

$$\lim_{t \to +\infty} \frac{\varphi(t)}{t} = \lim_{x \to +\infty} \varphi'(t) = c$$

Hence

$$\frac{F(\varphi(t))}{t^n} = \frac{F(\varphi(t))}{(\varphi(t))^n} \cdot \left(\frac{\varphi(t)}{t}\right)^n \to \frac{\lambda c^n}{n!} \tag{10.4}$$

On the other hand it is easily seen that

$$\left(\frac{1}{\varphi'} D_t\right)^n (F \circ \varphi) = \sum_{k=0}^{n} D_t^{n-k}[\alpha_k(t) F(\varphi(t))]$$

where $\alpha_0 = (1/\varphi')^n$ and $\alpha_k \in o(t^{-k})$ as $t \to +\infty$, for $k = 1, \ldots, n$. Thus all terms in last sum tend to zero as $x \to +\infty$, except $D_t^n[\alpha_0(F \circ \varphi)]$, which, according to 10.4 tends to λ.

This criterion and the corresponding ones for the cases when $x \to -\infty$, $t \to -\infty$, etc. lead to the following substitution rule for integrals:

10.5. Corollary. *Let f be a distribution integrable on* $\mathbf{R}$ *and* φ *a* C^∞ *mapping of* $\mathbf{R}$ *onto* $\mathbf{R}$ *such that:*

(j) $\varphi'(t)$ *is* $\neq 0$ *on* $\mathbf{R}$ *and tends to numbers* $\neq 0$ *as* $t\to+\infty$ *and as* $t\to-\infty$ *(in the ordinary sense)*

(jj) $\varphi^{(k+1)}\in o(t^{-k})$ *as* $t\to\infty$ *(in the ordinary sense) for all* $k=1,2,\ldots$

Then $f(\varphi(t))$ *is integrable on* $\mathbf{R}$ *and*

$$\int_{\mathbf{R}} f(x)dx = \int_{\mathbf{R}} f(\varphi(t))\,|\,\varphi'(t)\,|\,dt\ .$$

This rule is an immediate consequence of theorem 10.3 and its analogues, applied to a primitive of f. Observe that in the case $\varphi'(t)<0$:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{+\infty}^{-\infty} f(\varphi(t))\varphi'(t)dt = -\int_{-\infty}^{+\infty} f(\varphi(t))\varphi'(t)dt\ .$$

In particular, 10.5 applies in the elementary cases when $x=t+a$ or $x=ct$, with $a\in\mathbf{R}$ and $c\in\mathbf{C}$. Then we have

10.6
$$\int_{\mathbf{R}} f(x)dx = |\,c\,|\int_{\mathbf{R}} f(cx)\,dx$$

10.7
$$\int_{\mathbf{R}} f(x+a)dx = \int_{\mathbf{R}} f(x)dx\ .$$

The last formula can be expressed by saying that the integral is invariant under translations.

More refined criteria can be obtained, by using the concept of measure, as we did for multiplication:

Remember that, if μ is a measure on an open interval I, the *total variation of* μ in a bounded interval J such that $J\subset I$ is defined to be the supremum of the sums $S_P=\Sigma_1^p|\,\mu(J_i)\,|$, for all finite partitions P of J in intervals $J_1,\ldots,J_p$. We shall denote by $|\,\mu\,|(J)$ the total variation of μ in J; as is well known, $|\,\mu\,|$ is

again a measure on $\mathbf{R}$ (the *modulus of* μ) such that: (i) if $\mu \in \dot{L}(I)$, then $|\mu|$ is the modulus of the *function* μ in the ordinary sense; (ii) $|\varphi\mu|=|\varphi||\mu|$ for all $\varphi \in C(I)$. On the other hand, if μ and ν are two measures on I, we write $\mu \leqslant \nu$, iff $\mu(J) \leqslant \nu(J)$ for each bounded interval J such that $\bar{J} \subset I$.

Suppose I unbounded on the rigth. A measure μ on I is said to be *bounded on the right*, iff there exist two numbers x_0 and k such that $|\mu| \leqslant k$ for $x > x_0$, i. e. $|\mu|(J) \leqslant k|J|$ for all bounded intervals $J \subset [x_0, +\infty[$. On the other hand, we say that μ *converges to a number* c *as* $x \to +\infty$, iff for every $\varepsilon > 0$ there exists a real x_0 such that $|\mu - c| < \varepsilon$ for $x < x_0$. It is readily seen that these concepts coincide with the classical ones if μ turns out to be a function. Besides, the preceding lemmas for the «o» and «O» symbols keep true, if f is a measure.

These remarks suggest the following refinement of the concept of convergence for distributions:

10.8. DEFINITION. Let $f \in C_\infty(I)$, $n \in \mathbf{N}_0$ and $\lambda \in \mathbf{C}$. We write $f \xrightarrow[n]{} \lambda$ as $x \to +\infty$, iff there exist a real x_0 and a measure F such that $f = D^n F$ for $x > x_0$ and $F(x)/x^n \to \lambda/n!$ in the measure sense as $x \to \infty$.

On the other hand, if $\varphi \in C^\infty(I)$, we shall write $f \in o_n(\varphi)$ as $x \to +\infty$, iff there exist x_0 and f_0 such that $f = \varphi f_0$ for $x > x_0$ and $f_0 \xrightarrow[n]{} 0$ as $x \to +\infty$.

The expression «$f \in O_n(\varphi)$» can be analogously defined and the «dual» concepts of the preceding ones can be introduced as follows:

10.9. DEFINITION. Let $n \in \mathbf{N}_0$, $f \in C^n(I)$ and $\lambda \in \mathbf{C}$. We write $f \xrightarrow{n} \lambda$ as $x \to +\infty$, iff f tends to λ and $f^{(k)} \in o(x^{-k})$ as $x \to +\infty$, for $k=1,\ldots,n$ (in the ordinary sense). We write $f \in o^n(\varphi)$ as $x \to +\infty$, iff there exist x_0 and f_0 such that $f = \varphi f_0$ for $x > x_0$ and $f_0 \xrightarrow{n} 0$ as $x \to +\infty$.

That being so, it is readily seen that:

10.10. *If* $f \xrightarrow[n]{} \lambda$ *as* $x \to +\infty$ *and* $g \xrightarrow{n} \mu$ *as* $x \to +\infty$ *then* $fg \xrightarrow[n]{} \lambda\mu$ *as* $x \to +\infty$.

10.11. *If* $f \in o_n(\varphi)$ *and* $g \in o^n(\psi)$ *on the right, then* $fg \in o_n(\varphi\psi)$ *on the right, and analogously for the* «O» *symbols.*

There are similar criteria, which extend theorem 10.3 and corollary 10.5, for change of variable.

§ 3. PARTIAL INTEGRALS AND MULTIPLE INTEGRALS

11. Partial limits for distributions of two variables

Let I and J be two intervals in $\mathbf{R}$, and suppose that J is unbounded on the right. Given two functions $f(x,y)$ and $g(x)$ respectively on $I\times J$ and I, $f(x,y)$ is said to *converge uniformly on I to $g(x)$ as $y\to+\infty$*, iff for every $\varepsilon>0$ there exists a $\eta\in J$ (independent of x), such that: $|f(x,y)-g(x)|<\varepsilon$ for all $y>\eta$ and $x\in I$. On the other hand, if α is any real, we write

$$f(x,y)\in o(y^\alpha) \qquad \textit{uniformly} \text{ on } I \text{ as } y\to+\infty\,,$$

iff $f(x,y)/y^\alpha\to 0$ uniformly on I as $y\to+\infty$.

Put for every $f\in C(I\times J)$:

$$\mathfrak{J}_x f(x,y)\equiv\int_{x_0}^{x} f(\xi,y)d\xi\ ,\quad \mathfrak{J}_y f(x,y)\equiv\int_{y_0}^{y} f(x,\eta)d\eta\ ,$$

where x_0 resp. y_0 is a fixed point, arbitrarily chosen in I resp. J. The following lemma is easily proved (cf. 5.3):

11.1. Lemma. *If $\alpha>-1$ and $f(x,y)\in o(y^\alpha)$ uniformly on I as $y\to+\infty$, then:*

$$\mathfrak{J}_x f\in o(y^\alpha) \quad \textit{and} \quad \mathfrak{J}_y f\in o(y^{\alpha+1}) \quad \textit{uniformly on } I \textit{ as } y\to+\infty\,.$$

This lemma leads to the following:

11.2. Definition. If $f\in C_\infty(I\times J)$ and $\alpha>-1$, we write:

$$f(x,y)\in o(y^\alpha) \ \textit{on}\ I\ \textit{as}\ y\to+\infty,$$

iff there exist $m,n\in\mathbf{N}_0$ and $F\in C(I\times J)$ such that:

(1) $f(x,y)=D_x^m D_y^n F(x,y)$.

(2) $F(x,y)\in o(y^{\alpha+n})$ uniformly on each compact interval $I^*\subset I$ as $y\to+\infty$ (1)

Applying this definition and the lemma, the following properties are easily shown (with $\alpha>-1$):

11.3. *If $f(x,y)\in o(y^\alpha)$ and $g(x,y)\in o(y^\alpha)$ on I as $y\to+\infty$, then, for all $\lambda,\mu\in\mathbf{C}$:*

$$\lambda f+\mu g\in o(y^\alpha) \quad \textit{on } I \text{ as } y\to+\infty$$

(1) A more general condition might be considered instead of (2), but this definition is quite sufficient for applications.

11.4. *If* $f(x,y) \in o(y^{\alpha})$ *on* I *as* $y \to +\infty$, *then*

$$D_x f(x,y) \in o(y^{\alpha}) \quad \text{on } I \text{ as } y \to +\infty .$$

11.5. *If* $f(x,y) \in o(y^{\alpha})$ *on* I *as* $y \to +\infty$ *and* $\varphi(x)$ *is multipliable by* $f(x,y)$, *then*

$$\varphi(x) f(x,y) \in o(y^{\alpha}) \quad , \quad \text{on } I \text{ as } y \to +\infty .$$

Consider now $g \in C_\infty(I)$ and $f \in C_\infty(I \times J)$; then:

11.6. Definition. We say that $f(x,y)$ *converges on* I *to* $g(x)$, if and only if $f(x,y) - g(x) \in o(1)$ on I *as* $y \to +\infty$.

The uniqueness property as well as the linearity property of convergence are in this case immediate consequences of 11.3. Then we can write

$$g(x) = \lim_{y \to +\infty} f(x,y) \quad \text{or} \quad g(x) = f(x, +\infty) \quad \text{on } I,$$

to express that $f(x,y) \to g(x)$ on I as $y \to +\infty$.

On the other hand, the following important property, *which does not hold in classical analysis*, is an immediate consequence of 11.4:

11.7. Differentiation Property. *If* $f(x,y) \to g(x)$ *on* I as $y \to +\infty$, *then* $D_x f(x,y) \to D_x g(x)$ *on* I *as* $y \to +\infty$, *that is:*

$$D_x \lim_{y \to +\infty} f(x,y) = \lim_{y \to +\infty} D_x f(x,y) \quad \text{on } I .$$

In turn from 11.5 follows:

11.7′ Multiplication Property. *If* $f(x,y) \to g(x)$ *on* I *as* $y \to +\infty$ *and* $\varphi(x)$ *is multipliable* by $f(x,y)$, *then*

$$\lim_{x \to +\infty} [\varphi(x) f(x,y)] = \varphi(x) \lim_{y \to +\infty} f(x,y) \quad \text{on } I .$$

Moreover, applying 11.7 and the linearity property, it is easily shown:

11.8. Substitution Property. *If* $f(x,y) \to g(x)$ *on* I *as* $y \to +\infty$, *and if* φ *is a mapping of an interval* I^* *into* I, *such that* $f(\varphi(t),y)$ *exists* (1), *then* $g(\varphi(t))$ *exists too and*

$$f(\varphi(t),y) \to g(\varphi(t)) \quad \text{on } I^* \text{ as } y \to +\infty .$$

(1) The change of variables for distributions of several variables is defined according to criteria similar to those adopted for the case of one variable.

This substitution rule concerns the *parameter* x. Substitution rules concerning the *converging variable* y can be easily found, as generalization of the criteria given in 10.

The «O» symbol is extended to distributions $f(x,y)$ on $I\times J$, with respect to y, in the following way:

11.9. Definition. If $\alpha>-1$, we write

$$f(x,y)\in \mathrm{O}(y^{\alpha}) \quad \textit{on } I \textit{ as } y\to+\infty$$

iff there exist $m,n\in \mathbf{N}_0$ and $F\in \mathcal{C}(I\times J)$ such that:

(i) $f(x,y)=D_x^m D_y^n F(x,y)$;

(ii) for every compact interval $I^*\subset I$, there exists a number M such that

$$\frac{F(x,y)}{(1+|y|)^{\alpha+n}}\leqslant M \quad \text{on} \quad I^*\times J \text{ (}^1\text{)}.$$

More generally, if $\varphi\in \mathcal{C}^{\infty}(J)$, we write

$$f(x,y)\in \mathrm{O}(\varphi(y)) \quad \text{on } I \text{ as } y\to+\infty\,,$$

iff there exist a real y_0 and a distribution $f_0(x,y)$ such that $f(x,y)=\varphi(x)f_0(x,y)$ for $y>y_0$ and $f_0(x,y)\in \mathrm{O}(1)$ on I as $y\to+\infty$.

Besides, by the linearity property it is easily shown:

11.10. Differentiation Properties. *If α is any real and $f(x,y)\in \mathrm{O}(y^{\alpha})$ on I as $y\to+\infty$, then*

$$D_x f(x,y)\in \mathrm{O}(y^{\alpha}) \quad \text{and} \quad D_y f(x,y)\in \mathrm{O}(y^{\alpha-1}) \text{ on } I \text{ as } y\to+\infty\,.$$

Obviously all preceding considerations extend to the case when J is an interval unbounded on the right and $y\to+\infty$.

12. Partial integrals for distributions of two variables

Let I and J be any two intervals in $\mathbf{R}$, and $f(x,y)$ a distribution on $I\times J$. A distribution $\varphi(x,y)$ such that $D_y\varphi(x,y)=f(x,y)$ will be called a *(partial) primitive*

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(1) The choice of $(1+|y|)^{\alpha+n}$ instead of $y^{\alpha+n}$ is only to make the quocient continuous on $I^*\times J$.

of $f(x,y)$ with respect to y. On the other hand, a distribution $u(x,y)$ on $I \times J$ is said to be *independent* of y, iff it reduces to a distribution g of the only variable x, i. e, iff it is of the form $u(x,y) = D_x^m G(x)$ with $m \in \mathbf{N}_0$, $G \in \mathcal{C}(I)$.

12.1. LEMMA. *A distribution u on $I \times J$ is independent of x, iff $D_y u = 0$.*

Proof. It is readily seen that, if u is independent of y, then $D_y u = 0$. Suppose now conversely that $D_y u = 0$ and assume $u = D_x^m D_y^n U$, with $m, n \in \mathbf{N}_0$ and $U \in \mathcal{C}(I)$. Then $D_y u = D_x^m D_y^{n+1} U = 0$ and therefore (cf. 2, axiom 4) U must be of the form $U(x,y) = \sum_0^{m-1} x^i a_i(y) + \sum_0^n y^j b_j(x)$, with $a_i \in \mathcal{C}(J)$ and $b_j \in \mathcal{C}(I)$. Hence $u(x,y) = D_x^m D_y^n U(x,y) = n!\, D_x^m b_n(x)$.

Now it is easily proved, as in the case of one variable:

12.2. THEOREM. *Every distribution f on $I \times J$ has infinitely many primitives with respect to y and two such primitives differ by a distribution independent of x.*

We are able to define, in a natural way, the concept of *partial* (or *parametric) integral* of a distribution $f(x,y)$. It will be sufficient to consider integrals on $\mathbf{R}$. Let I be any interval in $\mathbf{R}$ and $f \in \mathcal{C}_\infty(I \times \mathbf{R})$; then:

12.3. DEFINITION. The integral $\int_{\mathbf{R}} f(x,y)dy$ is said to be *convergent on I*, iff there exists a primitive φ of f with respect to y which is convergent on I as $y \to +\infty$. Then we write

$$\int_{\mathbf{R}} f(x,y)dy = \varphi(x,+\infty) - \varphi(x,-\infty) \quad \text{on} \quad I\,.$$

From 11.2 follows at once the uniqueness of the partial integral. From the properties of partial limits we can deduce the *linearity property for partial integrals* as well as the following properties:

12.4. DIFFERENCIATION PROPERTY. *If $\int_{\mathbf{R}} f(x,y)dy$ is convergent on I, so is $\int_{\mathbf{R}} f'_x(x,y)dy$ and*

$$D_x \int_{\mathbf{R}} f(x,y)dy = \int_{\mathbf{R}} D_x f(x,y)dy \quad \text{on} \quad I\,.$$

12.5 SUBSTITUTION PROPERTY. *If* $\int_{\mathbf{R}} f(x,y)dy = g(x)$ *on* I *and if* φ *is any continuous mapping of an interval* I^* *into* I *such that* $f(\varphi(t),y)$ *exists, then*

$$\int_{\mathbf{R}} f(\varphi(t),y)dy = g(\varphi(t)) \quad \text{on} \quad I^* .$$

As for substitutions concerning the integration variable y, the criteria established in 10 can be easily extended to partial integrals. In particular, we have for all $h \in \mathbf{R}$:

12.6. $$\int_{\mathbf{R}} f(x,y+h)dy = \int_{\mathbf{R}} f(x,y)dy$$

12.7. $$\int_{\mathbf{R}} f(x,hy)dy = |h| \int_{\mathbf{R}} f(x,y)dy .$$

Applying lemma 12.1, criterion 7.7 can be also extended to partial integrals:

12.8. THEOREM. *If for any compact interval* $I^* \subset I$ *there exists a compact interval* K *such that* $f(x,y)=0$ *on* $I^* \times (\mathbf{R}-K)$, *then* $\int_{\mathbf{R}} f(x,y)dy$ *is convergent on* I.

Finally the following extensions of 9.1 and 9.2 are easily proved:

12.9. THEOREM. *If* $\int_{\mathbf{R}} f(x,y)dy$ *is convergent on* I, *then* $f \in \mathrm{O}(y^{-1})$ *on* I *as* $y \to \infty$. *On the other hand, if there exists* $\alpha < -1$ *such that* $f \in \mathrm{O}(y^{\alpha})$ *on* I *as* $y \to \infty$, *then* $\int_{\mathbf{R}} f(x,y)dy$ *is convergent on* I.

REMARKS—I. If $f(x,y)$ is a function, then for the convergence of $\int_{\mathbf{R}} f(x,y)dy$ on I in distributional sense it is not sufficient (neither necessary) that the integral be convergent *for each* $x \in I$. Obviously, a sufficient condition is that the integral be uniformly convergent on each compact set contained in I. More generally, it can be proved that, if f is summable on each set $I^* \times \mathbf{R}$, where I^* is a compact

interval contained in I, then $\int_{\mathbf{R}} f(x,y)dy$ is convergent on I in distributional sense.

II. The differentiation property can be associated with the linearity property in a more general property. Let $p(D)$ be a *derivation polynomial*, that is an operator of the form $p(D)=\sum_1^n a_k D^k$, with $a_1,\ldots,a_n \in \mathbf{C}$. Then we have

$$p(D_x)\int_{\mathbf{R}} f(x,y)dy = \int_{\mathbf{R}} p(D_x)f(x,y)dy \quad \text{on} \quad I\,,$$

whenever the first integral is convergent on I.

EXAMPLE. The preceding remarks offer a simple justification of formula 7.6. Observe that in the usual sense:

12.10. $$\int_{\mathbf{R}} \frac{e^{ixy}}{1+y^2}\,dy = \pi e^{-|x|} \qquad \textit{for each } x \in \mathbf{R}\,.$$

This can be easily found by the *method of residues*. Besides, as x,y are real variables, we have $|e^{ixy}|=1$ and

$$\left|\frac{e^{ixy}}{1+y^2}\right| \leqslant \frac{1}{1+y^2} \qquad \text{for all } x,y \in \mathbf{R}\,.$$

Thus the integral 12.10 is dominated, for all $x \in \mathbf{R}$, by the integral $\int_{\mathbf{R}} (1+y^2)^{-1}dy$, which is obviously convergent. Hence, according to the Weierstrass test, the first integral is *uniformly convergent on* $\mathbf{R}$ *and therefore convergent on* $\mathbf{R}$ *in distributional sense.* Consequently

$$(1-D_x^2)\int_{\mathbf{R}} \frac{e^{ixy}}{1+y^2}\,dy = \int_{\mathbf{R}} \frac{(1-D_x^2)\,e^{ixy}}{1+y^2}\,dy = \int_{\mathbf{R}} e^{ixy}dy, \quad \text{on} \quad \mathbf{R}\,.$$

On the other hand, $D_x^2 e^{-|x|} = -D_x(e^{-|x|} sg\, x) = e^{-|x|} - 2e^{-|x|}\delta(x)$, so that $(1-D_x^2)\,e^{-|x|} = 2\,\delta(x)$. Hence from 12.10 follows:

12.11. $$\int_{\mathbf{R}} e^{ixy}dy = 2\pi\delta(x) \quad \text{on} \quad \mathbf{R}\,.$$

13. Multiple integrals (on $\mathbf{R}^n$)

Let f be a distribution on $\mathbf{R}^n$ and λ any complex number. We say that $f(x)$ converges to λ as $x \to +\infty_n$, iff there exist $r \in \mathbf{N}_0^n$ and $F \in \mathcal{C}(\mathbf{R}^n)$ such that $f = D^n F$ and

$$\frac{F(x)}{x_1^{r_1} \dots x_n^{r_n}} \longrightarrow \frac{\lambda}{r_1! \dots r_n!} \quad \text{as} \quad x_1 \to +\infty, \dots, x_n \to +\infty .$$

Then we write $\lambda = \lim\limits_{x \to +\infty_n} f(x)$ or $\lambda = f(+\infty_n)$. The uniqueness of the limit, as well as the linearity property can be proved by an argument similar to the one used in the case $r = 1$. The concept of convergence as $x \to -\infty_n$ is analogously defined.

On the other hand, every distribution φ such that $\bar{D}\varphi = f$ (where $\bar{D} = D_1 \dots D_n$) will be called a $\bar{1}$-*primitive of* f. It is easily seen that:

13.1. Theorem. *Every $f \in \mathcal{C}_\infty(\mathbf{R}^n)$ has infinitely many $\bar{1}$-primitives and two such primitives differ necessarily by a distribution of the form $\sum_1^n u_j$ where u_j is a distribution independent of x_j (that is, of the form $D^r U$, where U is a continuous function on $\mathbf{R}^n$ independent of x_j).*

So we shall write by definition

$$\int_x^{x'} f(\xi) d\xi = \bar{\Delta}_{x'-x} \varphi(x) \tag{13.2}$$

where φ is any $\bar{1}$-primitive of f and $\bar{\Delta}_h$ is the mixed difference operator $\Delta_{1h_1} \dots \Delta_{nh_n}$.

From 13.1 follows that formula 13.2 defines actually a distribution $\Phi(x, x')$ on $\mathbf{R}^{2n}$, independent of the choice of the primitive φ. To see this, it is sufficient to observe that $\Delta_{h_j} u_j = 0$ for every distribution u_j independent of j and every $h_j \in \mathbf{R}$.

13.3. Definition. A distribution f is said to be *integrable on* $\mathbf{R}^n$, iff $\int_x^{x'} f(\xi) d\xi$ is convergent as $(x, x') \to (-\infty_n, +\infty_n)$. Then we write

$$\int_{\mathbf{R}^n} f(x) dx = \lim_{\substack{x \to -\infty_n \\ x' \to +\infty_n}} \int_x^{x'} f(\xi) d\xi . \tag{13.4}$$

For example, if $n=2$:

$$\int_{\mathbf{R}^2} f(x_1,x_2)dx_1dx_2 = \varphi(+\infty,+\infty)-\varphi(+\infty,-\infty)-\varphi(-\infty,+\infty)+\varphi(-\infty,-\infty)$$

where φ is a $\bar{1}$-primitive of f with respect to x.

The integral of f on $\mathbf{R}^n$ can be also denoted by $\int_{\mathbf{R}^n} f$ or simply by $\int f$. Uniqueness and linearity properties are immediate consequences of the corresponding properties for limits. In order to obtain further criteria, it is convenient to introduce a suitable definition of a bounded distribution:

13.5. Definition. A distribution f is said to be *bounded on* $\mathbf{R}^n$, iff there exist $r\in\mathbf{N}_0^n$ and $F\in\mathcal{C}(\mathbf{R}^n)$ such that:

(i) $f=D^rF$

(ii) for every regular matrix A of order n the function $x_1^{-r_1}\dots x_n^{-r_n}F(Ax)$ is bounded on $\mathbf{R}^n$.

The linearity property of boundedness is easily proved.

13.6. Definition. Given $f\in\mathcal{C}_\infty(\mathbf{R}^n)$ and $\varphi\in\mathcal{C}^\infty(\mathbf{R}^n)$, we write $f\in O(\varphi)$ as $|x|\to\infty$ or simply $f\in O(\varphi)$, iff there exists a distribution f_0 bounded on $\mathbf{R}^n$ and a real $\varepsilon>0$, such that $f=\varphi f_0$ for $|x|>\varepsilon$.

The following generalisation of 9.1 is easily obtained:

13.7. Theorem. *If there exists $\alpha<-n$ such that $f\in O(|x|^\alpha)$, then f is integrable on $\mathbf{R}^n$.*

On the other hand:

13.8. Theorem. *Suppose $f\in O(|x|^\alpha)$ with $\alpha<-n$ and let φ be a $\mathcal{C}^\infty$ mapping of $\mathbf{R}^n$ onto itself such that:*

(i) *the Jacobian matrix $[D_i\varphi_j]$ of φ is regular on $\mathbf{R}^n$ and converges to a regular matrix as $|t|\to\infty$;*

(ii) *$D^rD_i\varphi_j\in o(t^r)$ for all $r\in\mathbf{N}_0^n$, $r\neq 0$, $i,j=1,\dots,n$;* (1)
then the classical substitution rule applies:

$$\int_{\mathbf{R}^n} f(x)dx = \int_{\mathbf{R}^n} f(\varphi(t))\left|J\begin{pmatrix}\varphi\\ t\end{pmatrix}\right|dt\,.$$

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(1) As far as functions are concerned it is understood that the stated conditions are to be taken in ordinary sense.

We are going to outline the proof only in the case when φ is a non degenerate *affine mapping*, that is a mapping of the form $\varphi(t) = c+Mt$, where c is any vector in $\mathbf{R}^n$ and M is a regular matrix of order n. This case may be taken as a model for the general case, since φ behaves *assimptotically* just as an affine mapping, according to (i).

Put $h(x) = (1+x_1^2+\dots x_n^2)^{1/2}$ and suppose $f \in \mathrm{O}(|x|^\alpha)$, with $\alpha < -n$. Then it is readily seen that $f \in \mathrm{o}(h^\alpha)$, i. e. there exist $r \in \mathbf{N}_0^n$ and $F \in \mathcal{C}$, such that $f = h^\alpha . D^r F$, with $x_1^{-r_1} \dots x_n^{-r_n} F(Ax)$ bounded on $\mathbf{R}^n$ for every regular matrix A of order n. In such conditions it is easily found:

$$\int f(x)dx = (-1)^{\|r\|} \int h^{(r)}(x)F(x)dx, \quad \textit{where} \quad \|r\| = r_1+\dots+r_n \,.$$

Now

$$\int h^{(r)}(x)F(x)dx = \int h^{(r)}(\varphi(t))F(\varphi(t))\,|det\ M|\,dt$$

and it can be seen, without difficulty, that the last integral is just equal to

$$(-1)^{\|r\|} . \int f(h(t))\,|det\ M|\,dt \,.$$

14. Partial and multiple integrals

Let us consider a distribution $f(x,y)$ on $\mathbf{R}^{m+n}$ with $x \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$ $(m,n = 1,2,\dots)$. The concept of partial integral $\int_{\mathbf{R}^n} f(x,y)dy$ can be easily defined as a generalization of preceding concepts of partial integral and multiple integral, with similar properties. But there is also a new property:

14.1. THEOREM. *If $f(x,y)$ is integrable on $\mathbf{R}^{m+n}$ and if, in addition, the integral $\int_{\mathbf{R}^n} f(x,y)dy$ is convergent on $\mathbf{R}^m$, then*

$$\int_{\mathbf{R}^{m+n}} f(x,y)dxdy = \int_{\mathbf{R}^m} \left[\int_{\mathbf{R}^n} f(x,y)dy \right] dx \,.$$

This is a consequence of a property for limits that we can state as follows:

14.2. THEOREM. *If $f(x,y)$ is convergent as $(x,y)\to(+\infty_m,+\infty_n)$ and if in addition $f(x,y)$ is convergent on $\mathbf{R}^m$ as $y\to+\infty_n$, then*

$$\lim_{\substack{x\to+\infty_m\\ y\to+\infty_n}} f(x,y) = \lim_{x\to+\infty_m}\ [\lim_{y\to+\infty_n} f(x,y)]\,.$$

It is sufficient to prove this rule in the case $m=n=1$. Suppose that the hypothesis holds. Then there exist four integers p,q,r,s, two functions $F_1,F_2\in C(\mathbf{R}^2)$, a function $G\in C(\mathbf{R})$ and a number λ, such that $f=D_x^pD_y^qF_1=D_x^rD_y^sF_2$ and

$$\text{(i)}\qquad \frac{F_1(x,y)}{x^py^q}\to\frac{\lambda}{p!q!}\ ,\quad \text{as}\quad (x,y)\to(+\infty,+\infty)$$

$$\text{(ii)}\qquad \frac{F_2(x,y)}{y^s}\to\frac{G(x)}{s!}\ ,\quad \text{uniformly on each compact set in } \mathbf{R} \text{ as } y\to+\infty$$

We can assume $p=r, q=s$. Take $\varepsilon>0$. Then according to (i) there exist $a,b>0$ such that

$$14.3.\qquad \left|\frac{F_1(x,y)}{x^py^q}-\frac{\lambda}{p!q!}\right|<\varepsilon\quad,\quad \text{for } x>a\ ,\ y>b\,.$$

Take now p arbitrary points $x_j>a$, q arbitrary points $y_k>b$ and consider two pseudo-polynomials $P_1(x,y), P_2(x,y)$ of degree (p,q) sucht hat F_1-P_1 and F_2-P_2 vanish on the lines $x=x_j, y=y_k$. Then, if we put $F_0=F_1-P_1$, we have again $F_0=F_2-P_2$, so that $f=D_x^pD_y^qF_0$. Now, taking 14.3 into account and remembering that the coefficients of the pseudo-polynomials P_1,P_2 are obtained as linear combinations, respectively, of the values of $F_1(x,y)$ and $F_2(x,y)$ on the lines $x=x_j, y=y_k$, it is easily seen that (i) and (ii) are again satisfied, replacing F_1 and F_2 by F_0 and G by $G^*=G+P$, where P is a polynomial in x of degree $<p$. Hence from 14.3 follows, with F_0 in the place of F_1 and taking the limit as $y\to+\infty$:

$$\left|\frac{G^*(x)}{x^p}-\frac{\lambda}{p!}\right|\leqslant q!\,\varepsilon\ ,\qquad \text{for}\quad x>a\,.$$

The number ε being arbitrary, this implies that $G^*(x)/x^p\to\lambda/p!$ as $x\to+\infty$, which means that $\lambda=\lim\limits_{x\to+\infty}\ \lim\limits_{y\to+\infty} f(x,y)$.

More generally:

14.4. *If* $f(x,y,z)$ *is a distribution on* $\mathbf{R}^{m+n+p}$ *with* $x\in\mathbf{R}^m$, $y\in\mathbf{R}^n$, $z\in\mathbf{R}^p$, *such that* $\int_{\mathbf{R}^{n+p}} f(x,y,z)dydz$ *is convergent on* $\mathbf{R}^m$ *and* $\int_{\mathbf{R}^p} f(x,y,z)dz$ *is convergent on* $\mathbf{R}^{m+n}$, *then*

$$\int_{\mathbf{R}^{n+p}} f(x,y,z)dy\,dz = \int_{\mathbf{R}^n}\Big[\int_{\mathbf{R}^p} f(x,y,z)dz\Big]dy\,.$$

§ 4. CONVOLUTION

15. Convolution of two distributions on R

Consider two distributions $f=D^mF$ and $g=D^nG$, where $F,G\in\mathcal{C}(\mathbf{R})$. Then we have

$$f(x-t)=D_x^mF(x-t)=(-1)^mD_t^mF(x-t)\ ,$$

so that, for every $k=0,1,\ldots$,

$$D_t^kf(x-t)=(-1)^kD_x^kf(x-t)\ .$$

This suggests to write by definition

$$f(x-t)g(t)=f(x-t)D_t^nG(t)=\sum_{k=0}^{n}\binom{n}{k}D_t^{n-k}[G(t)D_x^kf(x-t)]\ ,$$

with $G(t)D_x^kf(x-t)=D_x^{m+k}[F(x-t)G(t)]$, that is

15.1. $$f(x-t)g(t)=\sum^{n}\binom{n}{k}D_x^{m+k}D_t^{n-k}[F(x-t)G(t)]\ .$$

It is easily seen that the «product» $f(x-t)g(t)$ does not depend on the representation of the distributions f and g. This can be proved as in the simpler case of the product of a $\mathcal{C}^n$ function φ by a distribution g of the form $g=D^nG$ with $G\in\mathcal{C}(\mathbf{R})$. The analogy between these two situations come from the well--known proposition (which is not needed for the present developments):

The mapping $t\to f(x-t)$ *of* $\mathbf{R}$ *into the topological linear space* $\mathcal{C}_\infty(\mathbf{R})$ *is infinitely differentiable.*

Consider now the expression $f(x-t)g(t-y)$. We have two possible interpretations:

15.2. $$\begin{cases} f(x-t)g(t-y) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} D_x^{m+k} D_t^{n-k}[F(x-t)G(t-y)] \\ f(x-t)g(t-y) = \displaystyle\sum_{k=0}^{m} \binom{m}{k} (-1)^k D_t^{m-k} D_y^{n+k}[F(x-t)G(t-y)]. \end{cases}$$

Now:

15.3. *The right members of the formulas* 15.2 *represent the same distribution.*

A direct proof of this proposition does not seem to be easy. On the contrary, a very simple proof can be found, remembering that the functions F and G can be aproached by two sequences (F_ν) and (G_ν) of C^∞ functions, converging uniformly on each compact interval, so that $D^m F_\nu \to f$ and $D^n G_\nu \to g$ as $\nu \to \infty$, in the distributional sense [1].

15.4. Definition. If the integral $\int_{\mathbf{R}} F(x-t)g(t)dt$ is convergent on $\mathbf{R}$, the distribution

$$h(x) = \int_{\mathbf{R}} f(x-t)g(t)dt$$

is called the *convolution* of f and g and denoted by $f*g$.

From this definition, taking into account the linearity property of the partial integral, as well as 15.1, follows immediately that the *convolution is bilinear*, that is, we have:

15.5. $$(\alpha f_1 + \beta f_2)*g = \alpha(f_1*g) + \beta(f_2*g) \quad , \quad \forall\, \alpha, \beta \in \mathbf{C}$$

whenever f_1*g and f_2*g exist; and analogously for the right side.

15.6. Commutative Law. *If $f*g$ exists, $g*f$ exists too and $f*g = g*f$.*

Proof. Suppose that $f*g$ exists and put $h = f*g$, that is $h(x) = \int_{\mathbf{R}} f(x-t)g(t)dt$.

Then for each $y \in \mathbf{R}$ we have

$$h(x-y) = \int_{\mathbf{R}} f(x-y-t)g(t)dt$$

---

[1] This is a well-known result of the Schwartz's theory, which can be established directly without difficulty.

and it is obvious that the last integral is still convergent with respect to (x,y) on $\mathbf{R}^2$. On the other hand, for each $y \in \mathbf{R}$, we may perform on this integral the substitution $t = u-y$, which gives:

$$h(x-y) = \int_{\mathbf{R}} f(x-u)g(u-y)du$$

Now, taking 15.3 into account, it can be seen that the last integral is also convergent with respect to y *for each* $x \in \mathbf{R}$. In particular for $x=0$ we have

$$h(-y) = \int_{\mathbf{R}} f(-u)g(u-y)du$$

Hence by the substitution $y = -x$, $u = -t$:

$$h(x) = \int_{\mathbf{R}} g(x-t)f(t)dt \quad \text{that is} \quad h = g*f \, .$$

In the general case the convolution is not associative. But the following criterion can be used in several cases:

15.7. *Id* f,g,h *are distributions on* $\mathbf{R}$ *such that* $f*g$ *and* $g*h$ *exist and* $\int_{\mathbf{R}^2} f(x-y)g(y-t)h(t)dydt$ *is convergent on* $\mathbf{R}$, *then*

$$(f*g)*h = f*(g*h) = \int_{\mathbf{R}^2} f(\hat{x}-y)g(y-t)h(t)dydt \, .$$

This is an immediate consequence of 14.4.

In turn, from the differentiation and substitution properties for partial integrals and from 15.6 follows immediately, taking definition 15.4 into account:

15.8. Differentiation Property. *If* $f*g$ *exists, then* $D(f*g)$ *exists too and*

$$D(f*g) = Df*g = f*Dg \, .$$

15.9. Translation Property. *If* $f*g$ *exists, then* $\tau_h(f*g)$ *exists for every* $h \in \mathbf{R}$ *and*

$$\tau_h(f*g) = (\tau_h f)*g = f*(\tau_h g) \, .$$

On the other hand:

15.10. *If* $f*g$ *and* $f*(xg)$ *exist, then* $(\hat{x}f)*g$ *exists too and*

$$\hat{x}(f*g) = (\hat{x}f)*g + f*(\hat{x}g) \, .$$

Proof. It is sufficient to observe that $(\hat{x}f)*g$ is given by

$$\int_{\mathbf{R}}(x-t)f(x-t)g(t)dt = x\int_{\mathbf{R}}f(x-t)g(t)dt - \int_{\mathbf{R}}f(x-t)tg(t)dt \; .$$

This important property shows that multiplication by x, with respect to convolution, behaves like a derivation operator with respect to multiplication.

Finally, we can analogously prove that:

15.11. *If $f*g$ exists, then*

$$e^{ax}(f*g) = (e^{ax}f)*(e^{ax}g) \; , \qquad \forall\, a \in \mathbf{C} \; .$$

16. Convolution of distributions whose support is bounded on the left and (or) on the right

We shall denote by $\mathcal{C}^0_\infty(\mathbf{R})$ or simply by $\mathcal{C}^0_\infty$ the vector space of all *distributions with bounded support on* $\mathbf{R}$.

16.1. Theorem. *The convolution $f*g$ exists whenever $f \in \mathcal{C}^0_\infty$ and $g \in \mathcal{C}_\infty$. Besides:*

(i) $f*(g*h) = (f*g)*h$, *whenever* $f, g \in \mathcal{C}^0_\infty$, $h \in \mathcal{C}_\infty$.

(ii) $\delta * f = f$, *for every* $f \in \mathcal{C}^0_\infty$.

Proof. *a*) Suppose $f \in \mathcal{C}^0_\infty$, $g \in \mathcal{C}_\infty$. Then there exists a bounded interval I such that $g(x-t)f(t)$ vanishes for $t \in \mathbf{R}-I$. Hence $\int_{\mathbf{R}} g(x-t)f(t)dt$ is convergent on $\mathbf{R}$ and gives $f*g$.

b) Suppose $f, g \in \mathcal{C}^0_\infty$, $h \in \mathcal{C}_\infty$. Then by an argument similar to the preceding it is shown that the integral $\int_{\mathbf{R}^2} h(x-y)g(y-t)h(t)dydt$ is convergent on $\mathbf{R}$ and this, according to 15.7, implies (i),

c) Consider $f = D^n F$, where $F \in \mathcal{C}(\mathbf{R})$, and put $F_1 = FH$, $F_2 = F - F_1$. Now

$$H*F_1 = \int_0^{+\infty} H(x-t)F_1(t)dt = \int_0^x F_1(t)dt \; .$$

Hence $\delta * D^n F_1 = D^{n+1}(H * F_1) = D^n F_1$. It is seen analogously that $\delta * D^n F_2 = D^n F_2$, so that $\delta * f = D^n F_1 + D^n F_2 = f$.

This theorem along with 15.5 and 15.6, can be expressed by saying:

16.2. *The space C^0_∞ is a commutative algebra under the convolution and C_∞ is a module over that algebra, having δ as the unity element.*

Property (ii) in 16.1 can be expressed explicitely by the important formula:

16.3. $$f(x) = \int_{\mathbf{R}} \delta(x-t) f(t)\, dt \qquad \text{(DIRAC'S FORMULA)}$$

We shall denote by C^+_∞ (resp. C^-_∞) the vector space of all distributions vanishing on the left (resp. on the right) of 0 and by $\tilde{C}^+_\infty$ (resp. $\tilde{C}^-_\infty$) the space of all distributions whose carrier is bounded on the left (resp. on the right).

16.4. *The space $\tilde{C}^+_\infty$ (resp. $\tilde{C}^-_\infty$) is an algebra under the convolution and C^+_∞ (resp. C^-_∞) is a subalgebra of $\tilde{C}^+_\infty$ (resp. $\tilde{C}^+_\infty$).*

In fact, if $f, g \in \tilde{C}^+_\infty$, there exists a real c such that f and g vanish for $x < c$. Then $f(x-t)g(t)$ vanishes for $t < c$ and for $t > x - c$. Hence $\int_{\mathbf{R}} f(x-t)g(t)\,dt$ is convergent on $\mathbf{R}$ and vanishes for $x < 2c$. The remaining parts of the theorem are easily proved.

17. Convolution and orders of growth. Tempered distributions and rapidy decreasing distributions (on $\mathbf{R}$) [1]

Several criteria can be found, connecting convolution with orders of growth of distributions. One of those criteria is the following:

17.1. THEOREM. *Let α and β be two real numbers such that $\alpha \geqslant \beta$ and $\alpha + \beta + n < -1$, where n is the integer satisfying $0 \leqslant \alpha + n < 1$. On the other hand, let f and g be two continuous functions on $\mathbf{R}$ such that $f \in O(x^\alpha)$ and $g \in O(x^\beta)$. Then $f * g$ exists and $f * g \in O(x^\alpha)$* [2].

Proof. a) Suppose $\alpha + \beta < -1$ with $\alpha \geqslant 0$. Then, as $f \in O(x^\alpha)$, there exists a number M such that $|f(x)| \leqslant M(1 + |x|)^\alpha$ for all $x \in \mathbf{R}$. Hence

$$|f(x-t)| \leqslant M(1 + |x-t|)^\alpha \leqslant M(1 + |x|)^\alpha (1 + |t|)^\alpha, \quad \forall x, t \in \mathbf{R}$$

---

(1) See «Notes ajoutées pendant la correction des épreuves» after the «Résumé».

(2) It is understood: «in the ordinary sense as $x \to \infty$».

since $\alpha \geqslant 0$. So the integral $\int_{\mathbf{R}} f(x-t)g(t)dt$ is dominated by the integral $M(1+|x|)^{\alpha} \int_{\mathbf{R}} (1+|t|)^{\alpha}|g(t)|dt$. Since $g \in O(t^{\beta})$ and $\alpha+\beta<-1$, the last integral exists. Hence the first integral is uniformly convergent on each compact set in $\mathbf{R}$ and its modulus is $\leqslant MK(1+|x|^{\alpha})$, where $K=\int_{\mathbf{R}} (1+|t|)^{\alpha}|g(t)|dt$. Consequently $f*g \in O(x^{\alpha})$.

b) Let now n be the integer such that $0 \leqslant \alpha+n<1$ and suppose $\alpha+\beta+n<-1$, $\beta \leqslant \alpha$. Then $x^k f * x^{n-k} g$ exists and is $O(x^{\alpha+n})$ for $k=0,\ldots,n$, according to the previous conclusion. Hence (cf. 15.10):

$$x^n(f*g)=\sum_{k=0}^{n}\binom{n}{k}(x^k f * x^{n-k} g) \in O(x^{\alpha+n})$$

so that $f*g \in O(x^{\alpha})$.

17.2. Corollary. *Let α be a real <-2, $\mathcal{A}_{\alpha}$ the set of all continuous functions f on $\mathbf{R}$ such that $f \in O(x^{\alpha})$ as $x \to \infty$ and $\mathcal{B}_{\alpha}$ the set of all continuous functions g on $\mathbf{R}$ such that there exists a real β (depending on g) satisfying the conditions: $\alpha+\beta<-1$ and $g \in O(x^{\beta})$. Then $\mathcal{A}_{\alpha}$ is an algebra under the convolution and $\mathcal{B}_{\alpha}$ is a module over $\mathcal{A}_{\alpha}$.*

Proof. Applying to the theorem (changing the roles of α and β), it is readily seen that, $f*g$ exists and belongs to $\mathcal{B}_{\alpha}$ whenever $f \in \mathcal{A}_{\alpha}$ and $g \in \mathcal{B}_{\alpha}$; and that $f*g \in \mathcal{A}_{\alpha}$ whenever $f, g \in \mathcal{A}_{\alpha}$. So we have only to prove the associative law:

$$f*(g*h)=(f*g)*h, \qquad \forall f, g \in \mathcal{A}_{\alpha}, h \in \mathcal{B}_{\alpha}.$$

But this can be easily seen applying 15.7, as we did for 16.1.

17.3. Remark. The preceding theorem and corollary can be extended to locally summable functions, according to the following criterion (Fubini-Tonnelli's theorem): *If $f, g \in L(\mathbf{R})$,* then $\int_{\mathbf{R}} f(x-t)g(t)dt$ *is convergent almost everywhere in $\mathbf{R}$ and defines a functions $h \in L(\mathbf{R})$.* It is also true that the preceding integral is convergent in the mean on $\mathbf{R}$, so that $f*g$ exists in the distributional sense.

Applying 15.11 and taking the Fubini-Tonnelli's theorem into account, it is a simple matter to obtain the following generalization of 17.1:

17.4. THEOREM. *Let α, β be two real numbers satisfying the hypothesis of 17.1, α', β' two real numbers such that $\alpha'+\beta' \leqslant 0$ and f, g two locally summable functions such that $f \in O(x^\alpha e^{\alpha'|x|})$ and $g \in O(x^\beta e^{\beta'|x|})$. Then $f*g$ exists and $f*g \in O(x^\alpha e^{\gamma|x|})$, where $\gamma = \max(\alpha', \beta')$.*

For the proof it is convenient to consider f and g in the form $f=f_1+f_2$ and $g=g_1+g_2$, with $f_1, g_1 \in L^+$, $f_2, g_2 \in L^-$, remembering that $f_1*g_1 \in L^+$, $f_2*g_2 \in L^-$.

From 17.4 it is easily deduced a corresponding generalization of 17.2.

Now, applying the differentiation property, we can derive from the preceding criteria corresponding rules for distributions. For example let us denote by $\tilde{\mathcal{A}}_\alpha$, for every $\alpha < -2$, the set of all distributions of the form $f = \sum_{k=0}^{p} D^{n_k} F_k$, where $p, n_1, \ldots, n_p$ are arbitrary integers and F_k locally summable functions such that $F_k \in O(x^\alpha)$; and by $\tilde{\mathcal{B}}_\alpha$ the set of all distributions of the form $g = \sum_{k=0}^{q} D^{r_k} G_k$ where $q, r_1, \ldots, r_q$ are arbitrary integers and G_k locally summable functions such that $G_k \in O(x^\beta)$ with $\alpha+\beta < -1$ (β depending on g). Then it is easily seen that $\tilde{\mathcal{A}}_\alpha$ is an algebra under convolution and $\tilde{\mathcal{B}}_\alpha$ a module over $\tilde{\mathcal{A}}_\alpha$.

17.5. DEFINITION. A distribution on $\mathbf{R}$ is said to be *tempered (slowly increasing* or *of polynomial type)*, iff there is a real α such that $f \in O(x^\alpha)$ (in the distributional sense).

An equivalent definition to this is the following: *f is tempered, iff there exist two integers n, k and a function $F \in \mathcal{C}(\mathbf{R})$ such that $f = D^n F$ and $F \in O(x^k)$ in the ordinary sense.*

We shall denote by $\check{\mathcal{C}}_\infty(\mathbf{R})$ or simply by $\check{\mathcal{C}}_\infty$ the set of all tempered distributions. It is readily seen that $\check{\mathcal{C}}_\infty$ *is a vector space closed under the operator D.*

17.6. DEFINITION. A distribution f on $\mathbf{R}$ is said to be *rapidly decreasing*, iff, for every $\alpha < 0$, f can be represented in the form $f = \sum_{k=0}^{p} D^{n_k} F_k$, where $p, n_1, \ldots, n_p$ are arbitrary integers ($n_k \geqslant 0, p \geqslant 1$) and F_k continuous functions such that $F_k \in O(x^\alpha)$ in the ordinary sense.

We shall denote by $\hat{\mathcal{C}}_\infty$ the set of all rapidly decreasing distributions on $\mathbf{R}$. From the preceding results it is easily deduced the well known fact:

17.7. COROLLARY. *$\hat{\mathcal{C}}_\infty$ is an algebra under convolution and $\check{\mathcal{C}}_\infty$ a module over $\hat{\mathcal{C}}_\infty$.*

A similar result can be obtained, concerning the space $\check{\mathcal{C}}_\infty$ of all distributions of *exponential type* (that is, of the form $f = D^n F$, where F is a continuous function on $\mathbf{R}$ such that $F \in O(e^{\alpha|x|})$ for *some* real α and the space $\hat{\mathcal{C}}_\infty$ of all

exponentially decreasing distributions (that is, of the form $f=D^nF$ where F is a continuous function such that $F\in O(e^{\alpha|x|})$ for every $\alpha<0$).

Observe that $\mathcal{C}^0_\infty \subset \hat{\mathcal{C}}_\infty \subset \widehat{\mathcal{C}}_\infty \subset \breve{\mathcal{C}}_\infty \subset \hat{\mathcal{C}}_\infty \subset \mathcal{C}_\infty$.

18. Convolution on $\mathbf{R}^n$

The concept of convolution of distributions on $\mathbf{R}$ is readily extended to the case of distributions on $\mathbf{R}^n$ and all preceding properties of convolution can be generalized to this case (now we have to consider derivation operators, translation operators, etc. for several variables).

Theorem 16.1 is readily extended to distributions of several variables. As for theorem 16.4 it gives place to new possibilities in the case of n variables. Let Γ be any convex cone in $\mathbf{R}^n$ whose vertex is the origin and not reducing to an half plane. We shall denote by $\mathcal{C}_\infty[\Gamma]$ the set of all distributions on $\mathbf{R}^n$ vanishing outside some cone $a+\Gamma$, with $a\in\mathbf{R}^n$. Then it is easily seen that $\mathcal{C}_\infty(\Gamma)$ is an algebra *under convolution*; besides there exists a maxinal subspace $\mathcal{C}_\infty(\mathbf{R}^n)$ distinct from $\mathcal{C}_\infty(\Gamma)$, which is a module over $\mathcal{C}_\infty(\Gamma)$.

Finally the criteria given in 17. can be also extended to the case of n variables and connected between them and the preceding ones, according to the different variables (¹).

§ 5. FOURIER TRANSFORMATION

19. Fourier transformation for tempered distributions on $\mathbf{R}$

Let f be any distribution on $\mathbf{R}$. If the integral $\int_{\mathbf{R}} e^{ixy}f(y)dy$ is convergent on $\mathbf{R}$, then the distribution

19.1. $$g(x) = \int_{\mathbf{R}} e^{ixy}f(y)dy$$

is called the *Fourier transform* of f and we write

$$g(x) = \mathfrak{F}_{x|y}f(y) \qquad \text{or simply } g = \mathfrak{F}f\,.$$

(¹) See further results in «Notes ajoutées pendant la correction des épreuves» after the «Résumée».

The Fourier transform of f is often denoted by $\hat{f}$. For simplicity, we shall omit the subscript $\mathbf{R}$ in the integral symbol, when no confusion seems possible.

For example, we have seen that $\displaystyle\int \frac{e^{ixy}}{1+y^2}\,dy = \pi e^{-|x|}$ in the distributional sense. So we have

$$\mathfrak{F}_{x|y}\, \frac{1}{1+y^2} = \pi e^{-|x|} .$$

From here we have deduced $\displaystyle\int e^{ixy}dy = 2\pi\delta(x)$. Hence

19.2. $$\mathfrak{F}\,1 = 2\pi\delta .$$

On the other hand, since $e^{ixy}\delta(y)=\delta(y)$, it is readily seen that

19.3. $$\mathfrak{F}\,\delta = 1 .$$

We are going to establish in a direct way some fundamental properties of the Fourier transformation for distributions.

19.4 *If $\mathfrak{F}f$ and $\mathfrak{F}g$ exist, then $\mathfrak{F}(\lambda f+\mu g)$ exists too, for all $\lambda, \mu \in \mathbf{C}$ and we have $\mathfrak{F}(\lambda f+\mu g) = \lambda(\mathfrak{F}f) + \mu(\mathfrak{F}g)$.*

This is an immediate consequence of the linearity property of integrals.

19.5. *If $\mathfrak{F}f$ exists, then $\mathfrak{F}(Df)$ exists too and*

$$\mathfrak{F}(Df) = -i\hat{x}\,.(\mathfrak{F}f) \text{ (}^1\text{)}.$$

Proof. Observe that

$$e^{ixy}f'(y) = D_y[e^{ixy}f(y)] - ixe^{ixy}f(y) .$$

If $\mathfrak{F}f$ exists, i. e. if $\displaystyle\int e^{ixy}f(y)dy$ is convergent on $\mathbf{R}$, then $e^{ixy}f(y) \to 0$ on $\mathbf{R}$ as $y\to\infty$, and so

$$\int e^{ixy}f'(y)dy = -ix\int e^{ixy}f(y)dy$$

that is $\mathfrak{F}(Df) = -ix(\mathfrak{F}f)$.

---

(1) Here the sign ^ means that x is a dummy variable.

19.6. *If $\mathfrak{F}f$ exists, then $\mathfrak{F}(\hat{y}f)$ exists too and*

$$\mathfrak{F}(\hat{y}f) = -iD(\mathfrak{F}f).$$

Proof. If $\int e^{ixy}f(y)dy$ is convergent on $\mathbf{R}$, we have by the differentiation property

$$D\int e^{ixy}f(y)dy = i\int e^{ixy}yf(y)dy$$

that is $D(\mathfrak{F}f) = i\mathfrak{F}(\hat{y}f)$, hence $\mathfrak{F}(\hat{y}f) = -iD(\mathfrak{F}f)$.

Properties 19.4, 19.5 and 19.6 can be associated as follows:
If p is any polynomial, then:

19.7. $\mathfrak{F}[p(D)F] = p(-ix)(\mathfrak{F}f) \quad , \quad \mathfrak{F}[p(y)f] = p(-iD)(\mathfrak{F}f)$.

We shall now prove some existence criteria for Fourier transforms.

19.8. *If f is summable on $\mathbf{R}$, the $\mathfrak{F}f$ exists and is a bounded continuous function on $\mathbf{R}$.*

Proof. Suppose $f \in L(\mathbf{R})$. Since $|e^{ixy}f(y)| = |f(y)|$ for all $x,y \in \mathbf{R}$, the integral $\int |e^{ixy}f(y)|dy$ is dominated by the integral $\int |f(y)|dy$, which is convergent and independent of x. Hence the integral $\int e^{ixy}f(y)dy$ is uniformly convergent on $\mathbf{R}$, which implies that it is convergent on $\mathbf{R}$ in the distributional sense and it represents a continuous function $g(x)$ on $\mathbf{R}$. Now this function is bounded, since $|g(x)| \leqslant \int |f(y)|dy$ for all $x \in \mathbf{R}$.

We shall denote by $\mathcal{C}_b$ the space of all bounded continuous functions on $\mathbf{R}$. Remember that we have denoted by $\check{\mathcal{C}}_\infty$ the space of all tempered distributions on $\mathbf{R}$ (17.5). Now from 19.7 and 19.8 follows:

19.9. *If $f \in \check{\mathcal{C}}_\infty$, then $\mathfrak{F}f$ exists and $\mathfrak{F}f \in \check{\mathcal{C}}_\infty$.*

Proof. Suppose $f \in \check{\mathcal{C}}_\infty$. Then there exist $m,p \in \mathbf{N}_0$ and $F \in \mathcal{C}(\mathbf{R})$ such that $f = D^mF$ and $F \in O(x^p)$ (in the ordinary sense as $x \to \infty$). Put $\Phi = F/(1+i\hat{x})^{p+2}$. Then $f = D^m(1+i\hat{x})^{p+2}\Phi$ and $\Phi \in \mathcal{C}(\mathbf{R})$, $\Phi \in O(x^{-2})$. Therefore, by 19.7, $\mathfrak{F}\Phi$ exists and $\mathfrak{F}\Phi \in \mathcal{C}_b \subset \check{\mathcal{C}}_\infty$. Hence, by 19.6, $\mathfrak{F}f$ exists too and $\mathfrak{F}f = (-i\hat{x})^m(1+D)^{p+2}(\mathfrak{F}\Phi) \in \check{\mathcal{C}}_\infty$.

We next purpose to study the problem of the inversion of $\mathfrak{F}$. Observe that $\mathfrak{F}$ transforms 1 into $2\pi\delta$, δ into 1, D into multplication by $-i\hat{x}$ and multiplication by $\hat{x}$ into $-iD$. Hence, if $\mathfrak{F}^{-1}$ exists, it should transform δ into $1/2\pi$, 1 into δ, etc. and so we may expect that $\mathfrak{F}^{-1}$ is given by the formula:

19.10. $$f(y) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{-ixy} g(x)dx\ .$$

We shall provisionally denote by $\mathfrak{F}^*$ the transformation $g \to f$ defined by this formula. Observing that, in this case, $f(-y) = (2\pi)^{-1}\mathfrak{F}_{y|x}g(x)$, it is readily seen that $\mathfrak{F}^*$ has the required properties and that $\mathfrak{F}^*f$ exists for all $f \in \breve{C}_\infty$. Moreover:

19.11. *If* $f \in \breve{C}_\infty$ *and* $g = \mathfrak{F}f$, *then* $f = \mathfrak{F}^*g$. *Conversely, if* $g \in \breve{C}_\infty$ *and* $f = \mathfrak{F}^*g$ *then* $g = \mathfrak{F}f$.

Proof. Suppose $f \in \breve{C}_\infty$ and put $g = \mathfrak{F}f$, $h = \mathfrak{F}^*g$. Then

$$h(y) = \frac{1}{2\pi}\int e^{-ixy}\left[\int e^{ixy'}f(y')\,dy'\right]dx$$

and, *if it is allowed to interchange the integrations*, we find

$$h(y) = \frac{1}{2\pi}\int\left[\int e^{ix(y'-y)}dx\right]f(y')dy'$$

But $\int e^{ix(y'-y)}dx = 2\pi\delta(y'-y) = 2\pi\delta(y-y')$ (by 19.2) and so, by the Dirac's formula,

$$h(y) = \int \delta(y-y')f(y')dy' = f(y), \quad \text{that is} \quad \mathfrak{F}^*g = f\,.$$

It is shown analogously that, if $g \in \breve{C}_\infty$ and $f = \mathfrak{F}^*g$, then $g = \mathfrak{F}f$, *under the hypothesis that the integrations are interchangeable.* To prove this, it will be enough (according to 14.2) to show that the double integral

19.12. $$\iint e^{ix(y'-y)}f(y')dxdy'$$

is convergent on $\mathbf{R}$. This can be made, at first, in the case when $f \in L$; in this case the integral

19.13. $$\iint \frac{e^{-ixy}}{1+x^2}\, e^{ixy'}f(y')dxdy'$$

is uniformly convergent on $\mathbf{R}$, since for all $x,y,y' \in \mathbf{R}$,

$$\left| \frac{e^{ixy}}{1+x^2} \right| = \frac{1}{1+x^2} \quad , \qquad |e^{ixy'}f(y')| = |f(y')|$$

and the functions $(1+x^2)^{-1}$, $f(y')$ are summable on $\mathbf{R}$. Hence, by applying to 19.13 the operator $1-D_y^2$, it is seen that the integral 19.12 is convergent on $\mathbf{R}$ in the case when $f \in L$. Finally this conclusion can be extended for every $f \in \mathcal{C}_\infty$, by an argument similar to the proof of 19.9, observing that the integral 19.12 represents $\mathfrak{J}^*\mathfrak{J}$ and applying 19.7.

Hence we have proved that $\mathfrak{J}^* = \mathfrak{J}^{-1}$ in the case when $\mathfrak{J}$ is restricted to $\check{\mathcal{C}}_\infty$.

The preceding results can be summarized as follows:

19.14. Theorem. *$\mathfrak{J}$ is a* 1-1 *linear mapping of the space $\check{\mathcal{C}}_\infty$ onto itself, changing D into multiplication by $-i\hat{x}$, multiplication by $\hat{x}$ into $-iD$,* 1 *into* $2\pi\delta$ *and δ into* 1. *The inverse of $\mathfrak{J}$ is given by* 19.10.

20. Fourier transformation and convolution

The following theorem is well known:

20.1. Theorem. *If f and g are summable functions on $\mathbf{R}$, then $\mathfrak{J}$ transforms the convolution $f*g$ into the usual product of the continuous functions $\mathfrak{J}f$ and $\mathfrak{J}g$, that is*

$$\mathfrak{J}(f*g) = (\mathfrak{J}f)(\mathfrak{J}g) \, .$$

We shall give here the proof of this theorem. It is known that, if $f, g \in L$, then $f*g$ exists and $f*g \in L$ (Fubini-Tonnelli's theorem, 17.3). Put $\hat{f} = \mathfrak{J}f$, $\hat{g} = \mathfrak{J}g$. Then $\hat{f}, \hat{g} \in \mathcal{C}_b$ (th. 19.8) and

$$\hat{f}(x)\,\hat{g}(x) = \int_{\mathbf{R}} e^{ixu} f(u)\,du \, . \int_{\mathbf{R}} e^{ixv} g(v)\,dv$$

$$= \int_{\mathbf{R}^2} e^{ix(u+v)} f(u) g(v)\,du\,dv \, .$$

Put now $u+v=y, v=t$. Then $u=y-t, v=t$ ant it is readily seen that the transformation defined by these formulas has a Jacobian equal to 1 and maps $\mathbf{R}^2$ onto $\mathbf{R}^2$. Therefore

$$\hat{f}(u)\hat{g}(x)=\int_{\mathbf{R}^2} e^{ixy}f(y-t)g(t)dy\,dt$$

$$=\int_{\mathbf{R}} e^{ixy}\left[\int_{\mathbf{R}} f(y-t)g(t)dt\right]dy$$

so that $\hat{f}\hat{g}=\mathfrak{F}(f*g)$, that is $\mathfrak{F}(f*g)=(\mathfrak{F}f)(\mathfrak{F}g)$.

20.2. Corollary. *Let f,g be two distributions on $\mathbf{R}$ of the form $f=D^mF$, $g=D^nG$, where F, G are l.s. functions satisfying the condition: there exists an integer p such that $(1+i\hat{x})^pF$ and $(1+i\hat{x})^{-p}G$ are summable on $\mathbf{R}$. Then $\mathfrak{F}(f*g)=(\mathfrak{F}f)(\mathfrak{F}g)$.*

This is a consequence of the theorem, in conjunction with properties 15.8 and 15.10. The corollary can obviously be extended to distributions f,g each of the preceding form. Remembering now the definition of the space $\hat{\mathcal{C}}_\infty$ of all rapidly decreasing distributions (17.6), it is easily deduced from 20.2:

20.3. Corollary. *If $f\in\hat{\mathcal{C}}_\infty$ and $g\in\check{\mathcal{C}}_\infty$ then $\mathfrak{F}(f*g)=(\mathfrak{F}f)(\mathfrak{F}g)$.*

In order to characterize in a direct way the Fourier transforms of rapidly decreasing distributions, we shall at first establish two general criteria:

20.4. Theorem. *If f is a distribution of the form D^nF, where F is a locally summable function on $\mathbf{R}$ such that $F\in\mathrm{O}(x^{-p})$, p being an integer $\geqslant 0$, and if $\varphi=\mathfrak{F}f$, then φ is a $\mathcal{C}^{p-2}$ function such that $\varphi^{(k)}\in\mathrm{O}(x^n)$, for $k=0,\ldots,p-2$.*

Proof. Suppose that the hypothesis is satisfied and put $\Phi=\mathfrak{F}F$. Then $\varphi=(-i\hat{x})^n\Phi$ and, since $\hat{x}^kF\in\mathrm{O}(x^{-2})$ for $k=0,\ldots,p-2$, it follows, by 19.6 and 19.8, that $D^k\Phi\in\mathcal{C}_b$ for $k=0,\ldots,p-2$. Hence $\varphi\in\mathcal{C}^{p-2}$ and $\varphi^{(k)}\in\mathrm{O}(x^n)$ for $k=0,\ldots,p-2$.

20.5. Theorem. *If φ is a $\mathcal{C}^p$ function such that $\varphi^{(p)}\in\mathrm{O}(x^n)$, with $n,p\leqslant\mathbf{N}_0$, and if $f=\mathfrak{F}\varphi$, then f is of the form $f=(1+D)^{n+2}F$, where F is a continuous function such that $F\in\mathrm{O}(x^{-p})$.*

Proof. Suppose that the hypothesis is satisfied and put $\Phi=(1-i\,\hat{x})^{-n-2}\varphi$, $F=\mathfrak{F}\Phi$. Then $f=(1+D)^{n+2}F$. On the other hand, $\varphi^{(k)}\in\mathrm{O}(x^{n+p-k})$ for $k=0,\ldots,p$, and this implies $\Phi^{(p)}\in\mathrm{O}(x^{-2})$. Hence $\hat{x}^pF\in\mathcal{C}_b$ and so $F\in\mathrm{O}(x^{-p})$.

20.6. Definition. A tempered C^∞ function on $\mathbf{R}$ is a function $\varphi \in C^\infty(\mathbf{R})$, satisfying the following condition: for every $p = 0,1,\ldots$, there exists an integer n, such that $\varphi^{(p)} \in O(x^n)$ (in the ordinary sense as $x \to \infty$).

We shall denote by $\breve{C}^\infty$ the set of all tempered C^∞ functions on $\mathbf{R}$. It is easily seen that $\breve{C}^\infty$ is a vector subspace of $\breve{C}_\infty \cap C^\infty$, but it must be observed that $\breve{C}^\infty \neq \breve{C}_\infty \cap C^\infty$. Now from 20.4 and 9.2.5 follows:

20.7. Corollary. *The Fourier transformation maps the convolution algebra* $\hat{C}_\infty$ *onto the multiplication algebra* $\breve{C}^\infty$.

Proof. a) Suppose $f \in \hat{C}_\infty$. This implies that, for every $p = 0,1,\ldots$, f can be represented in the form $f = \sum_1^m D^{n_k} F_k$ where $F_k \in O(x^{-p-2})$ for $k = 1,\ldots,m$. Then, if $\varphi = \mathfrak{F} f$, it is easily seen, taking 20.4 into account, that $\varphi \in C^p$ and $\varphi^{(p)} \in O(x^\mu)$ where $\mu = \max\,(n_1,\ldots,n_k)$. Hence $\varphi \in \breve{C}^\infty$.

b) Suppose $\varphi \in \breve{C}^\infty$. Then, for every $p = 0,1,\ldots$, there exists n such that $\varphi^{(p)} \in O(x^n)$. Hence, if we put $f = \mathfrak{F}^{-1}\varphi$, we conclude applying 20.5 (which extends obviously to $\mathfrak{F}^{-1}$) that f is of the form $(1+D)^{n+2}F$, where F is a continuous function such that $F \in O(x^{-p})$. Hence $f \in \breve{C}_\infty$.

The remaining part the of corollary is an obvious consequence of 20.3.

21. Fourier transformation on $\mathbf{R}^n$

The Fourier transformation on $\mathbf{R}^n$ may be defined by the formula

$$g(x) = \int_{\mathbf{R}^n} e^{ixy} f(y)\,dy$$

where f is a distribution on $\mathbf{R}^n$ and $xy = \sum_1^n x_h y_h$. If this integral is convergent on $\mathbf{R}^n$ we write $g = \mathfrak{F} f$. A distribution f on $\mathbf{R}^n$ is said to be *tempered*, iff there exist two systems $r, s \in \mathbf{N}_0^n$ and a function $F \in C(\mathbf{R}^n)$ such that $f = D^r F$ and $F \in O(x_1^{s_1} \ldots x_n^{s_n})$ in the ordinary sense; then we write $f \in \breve{C}_\infty(\mathbf{R}^n)$ or simply $f \in \breve{C}_\infty$.

All preceding properties of the Fourier transformation can be extended to the present case, with the obvious modifications concerning the existence of n derivation operators and n coordinate functions, $\hat{x}_1,\ldots,\hat{x}_n$. Thus

$$\mathfrak{F}(D_k f) = (-ix_k)(\mathfrak{F} f), \quad \mathfrak{F}(\hat{x}_k f) = (-iD_k)\mathfrak{F}(f)$$

for all $f \in \breve{C}_\infty(\mathbf{R}^n)$ and $k = 1,\ldots,n$. Moreover, in the inversion formula, the coefficient $1/(2\pi)$ must be replaced by $1/(2\pi)^n$.

Observe that, if $f \in \check{C}_\infty(\mathbf{R}^n)$, we can define by the formula

$$g_k(x) = \int\limits_{\mathbf{R}} e^{ixy_k y_k} f(x) dx_k$$

the *Fourier transform of f with respect to x_k* (it is easily proved that this partial integral is then convergent on $\mathbf{R}^n$). Then we shall write $g_k = \mathfrak{F}_{x_k} f$ or simply $g_k = \mathfrak{F}_k f$ and it is easily seen that

$$\mathfrak{F} = \mathfrak{F}_1 \dots \mathfrak{F}_k \, .$$

But for the existence of $\mathfrak{F}_k f$ it is not necessary that $f \in \check{C}_\infty(\mathbf{R}^n)$: *it is sufficient that there exists an integer p such that*

$$f \in O(x_k^p) \quad \text{on} \quad \prod_{j \neq k} \mathbf{R}_{x_j} \quad \text{as} \quad x_k \to \infty \, .$$

Finally, applying theorem 13.8, it easily proved that *Fourier transformation on $\check{C}_\infty(\mathbf{R}^n)$ is invariant under isometric linear transformation of the $\mathbf{R}^n$-space with the usual norm.*

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RÉSUMÉ

1. Les symboles O et o pour des distributions

Pour fixer les idées, nous nous bornerons au cas où $x \to +\infty$. Soit I un intervalle ouvert de $\mathbf{R}$, non borné à droite, $I =]a, +\infty[$; si f er φ sont deux fonctions sur I, on écrit

$$f \in \mathrm{O}(\varphi) \qquad \text{lorsque } x \to +\infty$$

s'il existe un nombre réel x_0 et une fonction f_0 bornée pour $x > x_0$ tels que $f = \varphi f_0$ pour $x > x_0$. D'autre part on écrit

$$f \in \mathrm{o}(\varphi) \qquad \text{lorsque } x \to +\infty$$

s'il existe un nombre réel x_0 et une fonction f_0 tendant vers 0 lorsque $x \to +\infty$, tels que $f = \varphi f_0$ pour $x > x_0$.

En particulier, on peut avoir $\varphi = \hat{x}^\alpha$ où α est un nombre réel, c'est-à-dire $\varphi(x) \equiv x^\alpha$. Dans ce cas nous écrirons x^α au lieu de $\hat{x}^\alpha$, pour alléger les notations.

Soit maintenant c un point de I et posons $\mathfrak{I}f(x) = \int_c^x f(\xi)d\xi$ pour toute fonction f localement sommable sur I. Cela étant:

Lemmes 1 et 2. *Si α est nombre un réel > -1 et f une fonction localement sommable sur I telle que $f \in \mathrm{O}(x^\alpha)$ [resp. $f \in \mathrm{o}(x^\alpha)$] lorsque $x \to +\infty$, alors*

$$\mathfrak{I}f \in \mathrm{O}(x^{\alpha+1}) \ [\text{resp. } \mathfrak{I}f \in \mathrm{o}(x^{\alpha+1})] \ \textit{lorsque } x \to +\infty$$

Ces deux lemmes justifient les définitions suivantes:

Définitions 1 et 2. Si α est un nombre réel > -1 et f une distribution sur I, on écrit

$$f \in \mathrm{O}(x^\alpha) \ [\text{resp. } f \in \mathrm{o}(x^\alpha)] \quad \text{lorsque } x \to +\infty,$$

s'il existe un entier $n \geqslant 0$ et une fonction F localement sommable sur I, tels que $f = D^n F$ et $F \in O(x^{\alpha+n})$ [resp. $F \in o(x^{\alpha+n})$] lorsque $x \to +\infty$, au sens usuel. En particulier, si $f \in O(1)$ lorsque $x \to +\infty$, on dit que f est *bornée à droite*, et si $f \in o(1)$ lorsque $x \to +\infty$, on dit que f *tend vers* 0 *lorsque* $x \to +\infty$ (1). Cela étant, on peut encore élargir les notions de O et de o de la façon suivante:

DÉFINITIONS 1′ et 2′. Si φ est une fonction indéfiniment différentiable sur I et f une distribution sur I, on écrit

$$f \in O(\varphi) \ [\text{resp. } f \in o(\varphi)] \quad \text{lorsque } x \to +\infty,$$

s'il existe un réel x_0 et une distribution f_0 bornée à droite [resp. tendant vers zéro lorsque $x \to +\infty$] tels que $f = \varphi f_0$ pour $x > x_0$.

Les lemmes 1 et 2 permettent de voir que les définitions 1′ et 2′ sont équivalentes aux définitions 1 et 2 dans le cas où $\varphi = x^\alpha$ avec $\alpha > -1$. D'autre part, il est aisé de généraliser au cas des distributions plusieurs propriétés élémentaires des symboles O et o, parmi lesquelles la

PROPRIÉTÉ DE LINÉARITÉ. *Si* $f \in O(\varphi)$ *et* $g \in O(\varphi)$ *lorsque* $x \to +\infty$, et *si* $\lambda, \mu \in \mathbf{C}$, *alors*

$$\lambda f + \mu g \in O(\varphi) \quad \text{lorsque } x \to +\infty$$

(Et de même pour le symbole o*)*

Mais on obtient une propriété nouvelle:

Si α *est un nombre réel quelconque et* $f \in O(x^\alpha)$ *lorsque* $x \to +\infty$, *alors* $Df \in O(x^{\alpha-1})$ *lorsque* $x \to +\infty$; *et de même pour le symbole* o (cf. lemmes 1 et 2).

2. Limites et intégrales de distributions

Considérons encore, pour fixer les idées, le cas où $x \to +\infty$ et soit $I =]a, +\infty[$.

DÉFINITION 3. Si f est une distribution sur I et λ un nombre complexe, on écrit

$$f(x) \to \lambda \quad \text{lorsque } x \to +\infty$$

(1) Cette définition de «distribution bornée» est plus générale que celle donnée par L. SCHWARTZ.

si $f-\lambda \in o(1)$ lorsque $x \to +\infty$; ou, ce qui revient au même, s'il existe un entier $n \geqslant 0$ et une fonction F continue sur I, tels que $f = D^n F$ et $F(x)/x^n \to \lambda/n!$ lorsque $x \to +\infty$ (au sens usuel) [1].

On démontre aisément que, *si* $f \to \lambda$ *lorsque* $x \to +\infty$ *et* $f \to \mu$ *lorsque* $x \to +\infty$, *alors* $\lambda = \mu$. Cela justifie que l'on écrit ce cas $\lambda = \lim\limits_{x \to +\infty} f$ ou $\lambda = f(+\infty)$. D'autre part, on généralise plusieurs propriétés usuelles de la notion de limite. Observons encore que

Si une distribution f tend vers un nombre λ lorsque $x \to +\infty$, alors f est bornée à droite.

On définit d'une façon analogue les notions de limite d'une distribution lorsque $x \to -\infty$ ou lorsque $x \to a^+$, $x \to a^-$ or $x \to a$, où a est un réel quelconque.

De ces notions de limite découlent des notions correspondantes d'intégrale. Soit par exemple f une distribution sur $\mathbf{R}$. On sait qu'il existe au moins une primitive F de f; alors, on dit que f est *intégrable sur* $\mathbf{R}$ si F est convergente lorsque $x \to +\infty$ et lorsque $x \to -\infty$, et on écrit

$$\int_{-\infty}^{+\infty} f(x)dx = F(+\infty) - F(-\infty)\,.$$

L'intégrale de f sur $\mathbf{R}$ peut être aussi notée $\int_{\mathbf{R}} f$.

On démontre aisément l'unicité de l'intégrale, ainsi que sa linéarité, etc. Considérons par exemple l'intégrale $\int_{\mathbf{R}} e^{i\xi x} dx$, où ξ est un paramètre réel. Pour $\xi \neq 0$, une primitive de $e^{i\xi x}$ par rapport à x sera $e^{i\xi x}/i\xi$. À son tour

$$\frac{e^{i\xi x}}{i\xi} = D_x \frac{e^{i\xi x}}{(i\xi)^2} \quad \text{et} \quad \lim_{x \to \infty} \frac{e^{i\xi x}}{x} = 0$$

Donc, d'après les définitions précédentes:

$$e^{i\xi x}/i\xi \to 0 \qquad \text{lorsque} \quad x \to \infty\,,$$

et, par conséquent $\int_{\mathbf{R}} e^{i\xi x} dx = 0$, pour tout $\xi \neq 0$.

Si $\xi = 0$ l'intégrale $\int_{\mathbf{R}} e^{i\xi x} dx$ est divergente, même au sens des distributions. Ces résultats sont d'accord avec la formule

(1) $$\int_{\mathbf{R}} e^{i\xi x} dx = 2\pi\delta(\xi)$$

[1] Cette définition est plus générale que celle donnée par Mikusiński et Sikorski.

qui a été depuis longtemps introduite par les physiciens d'une façon heuristique. Mais cette formule ne pourra être justifiée complètement que par une généralisation convenable de la notion d'intégrale paramétrique, ce dont nous nous occupons plus loin.

Il est à remarquer que, d'après les définitions de limite d'une distribution, proposées, à notre connaissance, par d'autres auteurs, l'intégrale $\int_{\mathbf{R}} e^{i\xi x}dx$ n'est convergente pour *aucune valeur réelle* de ξ.

3. Critères de convergence pour des intégrales

Considérons, par exemple, le cas d'une intégrale sur $\mathbf{R}$. On a d'abord le critère (condition nécessaire de convergence), qui n'est pas valable dans le domaine classique:

Théorème 1. *Si f est une distribution intégrable sur $\mathbf{R}$, on a $f \in O(x^{-1})$ lorsque $x \to \infty$.*

D'autre part, on obtient la généralisation suivante d'un critère classique (condition suffisante de convergence):

Théorème 2. *S'il existe un réel $\alpha < -1$ tel que $f \in O(x^{\alpha})$ lorsque $x \to \infty$, alors f est intégrable sur $\mathbf{R}$.*

On peut encore établir des critères semblables pour d'autres types d'intervalles. Par exemple, considérons $I =]a, +\infty[$, avec $a \in \mathbf{R}$, et soit f une distribution sur I; alors:

Théorème 1′. *Si f est intégrable sur I, on a $f \in O(x^{-1})$ lorsque $x \to +\infty$ et $f \in O((x-a)^{-1})$ lorsque $x \to a^{+}$.*

Théorème 2′. *S'il existe deux réels α et β tels que $\alpha < -1$, $\beta > -1$, $f \in O(x^{\alpha})$ lorsque $x \to +\infty$ et $f \in O((x-a)^{\beta})$ lorsque $x \to a^{+}$, alors f est intégrable sur I.*

4. Limites partielles et intégrales paramétriques

Nous nous bornerons ici à un cas particulier qui servira de paradigme:

Définition 4. On dit qu'une distribution $f(x,y)$ sur $\mathbf{R}^2$ *tend vers une distribution* $g(x)$, *lorsque* $y \to +\infty$, s'il existe deux entiers $m, n \geqslant 0$ et deux

fonctions continues $F(x,y)$ et $G(x)$, respectivement sur $\mathbf{R}^2$ et sur $\mathbf{R}$, tels que: 1) $f(x,y) = D_x^m D_y^n F(x,y)$; 2) $g(x) = D_x^m G(x)$; 3) $\dfrac{F(x,y)}{y^n} \to \dfrac{G(x)}{n!}$ lorsque $y \to +\infty$, uniformément sur tout borné.

On démontre d'abord l'unicité de la limite, ce qui permet d'écrire

$$g(x) = \lim_{y\to+\infty} f(x,y) \quad \text{ou} \quad g(x) = f(x, +\infty)$$

Ensuite on établit des propriétés élémentaires telles que la linéarité, etc., et encore la *propriété nouvelle:*

$$D_x \lim_{y\to+\infty} f(x,y) = \lim_{y\to+\infty} D_x f(x,y)$$

La définition 4 et la définition correspondante pour le cas où $x \to -\infty$, permettent de définir l'intégrale paramétrique sur $\mathbf{R}$.

Définition 5. On dit que l'intégrale $\int_{\mathbf{R}} f(x,y)dy$ est convergente sur $\mathbf{R}$ (par rapport à x), si, étant donnée $F(x,y)$ telle que $D_y F = f$, les limites $F(x, +\infty)$ et $F(x, -\infty)$ existent. Alors on écrit $\int_{\mathbf{R}} f(x,y)dy = F(x, +\infty) - F(x, -\infty)$.

Evidemment, on aura non seulement l'unicité et la linéarité de l'intégrale paramétrique, mais aussi la *propriété nouvelle:*

$$D_x \int_{\mathbf{R}} f(x,y)dy = \int_{\mathbf{R}} D_x f(x,y)dy$$

En particulier, cela permet de démontrer la formule (1) par une méthode rigoureuse, dont la technique se rapprouche des méthodes heuristiques des physiciens.

5. Limites et intégrales multiples

Nous prendrons encore un cas particulier pour modèle:

Définition 6. On dit qu'une distribution $f(x,y)$ sur $\mathbf{R}^2$ tend vers un nombre λ lorsque $x \to +\infty$ et $y \to +\infty$, s'il existe deux entiers $m,n \geqslant 0$ et une fonction $F(x,y)$ continue sur $\mathbf{R}^2$ tels que: 1) $f(x,y) = D_x^m D_y^n F(x,y)$; 2) $\dfrac{F(x,y)}{x^m y^n}$ tend vers $\dfrac{\lambda}{m!n!}$ lorsque $x \to +\infty$ et $y \to +\infty$ (au sens usuel).

On peut écrire dans ce cas:

$$\lambda = \lim_{\substack{x \to +\infty \\ y \to +\infty}} f(x,y) = f(+\infty, +\infty)$$

l'unicité de la limite étant assurée (ainsi que sa linéarité).

On définit d'une façon analogue les limites $f(-\infty, -\infty)$, $f(-\infty, +\infty)$ et $f(+\infty, -\infty)$. Ces définitions conduisent à la définition suivante d'intégrale double:

Definition 7. On dit que f est *intégrable* sur $\mathbf{R}^2$, s'il existe une distribution F, telle que $f = D_x D_y F$, pour laquelle les limites $F(+\infty, +\infty)$, $F(-\infty, +\infty)$, $F(+\infty, -\infty)$, $F(-\infty, -\infty)$ existent. On écrit alors:

$$\int_{\mathbf{R}^2} f(x,y)dxdy = F(+\infty, +\infty) - F(-\infty, +\infty) - F(+\infty, -\infty) + F(-\infty, -\infty)$$

L'unicité et la linéarité de l'intégrale sont assurées. En outre:

Théorème 3. *Si l'intégrale* $\int_{\mathbf{R}} f(x,y)dy$ *est convergente par rapport à* x *sur* $\mathbf{R}$ *et si* f *est intégrable sur* $\mathbf{R}^2$, *alors*

$$\int_{\mathbf{R}^2} f(x,y)dxdy = \int_{\mathbf{R}} \left[\int_{\mathbf{R}} f(x,y)dy \right] dx$$

Toutefois l'intégrale double, telle que nous venons de la définir, n'est pas invariante pour les rotations. Mais cet inconvénient disparait dans les cas où s'applique le critère de convergence que nous indiquerons plus loin.

Dans le cas général, où il s'agit d'une distribution sur $\mathbf{R}^n$, on définit d'une façon semblable les limites multiples et l'intégrale sur $\mathbf{R}^n$

$$\int_{\mathbf{R}^n} f(x)dx = \int_{\mathbf{R}^n} f(x_1, \ldots, x_n)dx_1 \ldots dx_n$$

Plus généralement encore, étant donné une distribution $f(x,y) \equiv f(x_1, \ldots, x_m, y_1, \ldots, y_n)$ sur $\mathbf{R}^{m+n}$ on définit l'intégrale multiple-paramétrique

$$\int_{\mathbf{R}^n} f(x,y)dy = \int_{\mathbf{R}^n} f(x_1, \ldots, x_m, y_1, \ldots, y_n)dy_1 \ldots dy_n ,$$

en généralisant de façon triviale les définitions précédentes d'intégrale multiple et d'intégrale paramétrique.

Enfin, pour généraliser les critères de convergence aux intégrales multiples, il faut d'abord prolonger le symbole O aux distributions de n variables.

Définition 8. On dit qu'une distribution f est *bornée sur* $\mathbf{R}^n$, s'il existe un sistème r de n entiers $r_1,\ldots,r_n$ et une fonction F continue sur $\mathbf{R}^n$ tels que:

1) $f = D^r F$

2) pour toute matrice A régulière d'ordre n la fonction $x_1^{-r_1}\ldots x_n^{-r_n} F(Ax)$ est bornée sur $\mathbf{R}^n$.

Définition 9. Si φ est une fonction indéfiniment différentiable sur $\mathbf{R}^n$, on écrit

$$f \in \mathrm{O}(\varphi) \quad \text{sur } \mathbf{R}^n ,$$

s'il existe une distribution f_0 bornée sur $\mathbf{R}^n$ et un nombre réel $\varepsilon > 0$, tels que $f = \varphi f_0$ pour $|x| > \varepsilon$.

Cela posé, on démontre le critère suivant:

Théorème 4. *S'il existe un réel* $\alpha < -n$ *tel que* $f \in \mathrm{O}(|x|^\alpha)$, *où* $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$, *alors* f *est intégrable sur* $\mathbf{R}^n$.

6. Convolution et transformation de Fourier

Les notions et les résultats précédents permettent, en particulier, de construire une théorie directe de la convolution et des transformations de Fourier et de Laplace pour des distributions, qui se rapproche beaucoup des méthodes heuristiques des physiciens et qui offre des moyens d'obtenir des résultats essentiellement nouveaux.

D'abord deux distributions f et g sur $\mathbf{R}^n$ sont dites *composables*, si l'intégrale

$$\int_{\mathbf{R}^n} f(x-\xi) g(\xi) d\xi$$

est convergente, par rapport à x, sur $\mathbf{R}^n$; alors la distribution $h(x)$ définie par cette intégrale est nommée la *convolution* de f et g, et on écrit $h = f*g$. La convolution est bilinéaire et commutative; en outre, on a les propriétés suivantes:

$$D(f*g) = (Df)*g = f*(Dg)$$

$$x(f*g) = (xf)*g + f*(xg)$$

(on sous-entend que l'existence de deux des convolutions $x(f*g)$, $(xf)*g$ et $f*(xg)$ entraîne l'existence de la troisième).

Toutes ces propriétés permettent d'étudier plusieurs cas intéressants de couples de distributions composables et d'établir des conditions suffisantes pour que l'associativité se vérifie (dans le cas général, la convolution n'est pas associative).

D'autre part, si f est une distribution sur $\mathbf{R}^n$ telle que l'intégrale paramétrique

$$\int_{\mathbf{R}^n} e^{ix\xi} f(\xi)\,d\xi \qquad (x\xi = x_1\xi_1 + \ldots + x_n\xi_n)$$

est convergente sur $\mathbf{R}^n$, on définit une distribution $\varphi(x)$ sur $\mathbf{R}^n$ qui s'appelle la transformée de Fourier de f et on écrit $\varphi = \mathfrak{F}f$. Cela étant, on établit aussitôt les propriétés élémentaires de la transformation de Fourier, $\mathfrak{F}$, et on démontre aisément que $\mathfrak{F}$ définit une application biunivoque de l'espace des distributions tempérées sur lui-même, etc., etc.

1. **Sur la notion d'intégrale d'une distribution.** Récemment nous avons pu vérifier que la notion d'intégrale d'une distribution f d'une variable, telle que nous l'avons définie, coincide, dans le cas où f est une fonction localement sommable, avec une notion d'intégrale généralisée, que Du Bois-Reymond a introduit en 1887 (Journal de Crelle, vol. C, p. 356), en étendant aux intégrales les méthodes de sommation de Cesàro pour les séries. Voici l'idée de Du Bois-Reymond:

Soit r un entier $\geqslant 0$ et a un nombre réel quelcomque. Si l'on pose

$$F(x) = \int_a^x \frac{(x-t)^r}{r!} f(t)dt\,,$$

on a $F = \mathfrak{J}_a^{r+1} f$, donc $\mathfrak{J}_a f = \int_a^x f = D^r F$. Alors, si la fonction $F(x)/x^r$ tend vers une limite finie lorsque $x \to +\infty$, on dit que l'intégrale $\int_a^{+\infty} f(t)dt$ est *sommable* (C, r) et on écrit:

$$\int_a^{+\infty} f(t)dt = \lim_{x\to+\infty} \frac{r!\,F(x)}{x^r} = \lim_{x\to+\infty} \int_a^x \left(1-\frac{t}{x}\right)^r f(t)dt$$

Or, on voit aussitôt que, dans ce cas, la fonction f est intégrable au sens des distributions sur $[a, +\infty[$ et que son intégrale sur cette intervalle a pour valeur la limite ci-dessus indiquée.

Cette notion de sommabilité (C, r) peut évidemment s'étendre au cas où r est un nombre réel non-négatif quelconque.

Pour les détails, voir E. W. Hobson, «*The theory of fonctions of a real variable*», Dover Publications, Inc., New York, pp. 384-388, 737-741.

2. **Sur les critères d'existence de la convolution, faisant intervenir les ordres de croissance.** Les critères que nous avons indiqués au n.º 17 (Convolution and orders of growth) font intervenir seulement les notions usuelles d'ordre de croissance pour les fonctions. On obtient des critères beaucoup plus fins, si l'on emploie les notions d'ordre de croissance d'une distribution (cf. définitions 10.8 et 10.9). Par exemple:

Si f est g sont deux distributions sur $\mathbf{R}$ telles que $f \in \mathrm{O}_n(x^\alpha)$ et $g \in \mathrm{O}^n(x^\beta)$, et si α, β vérifient la condition $\alpha + \beta < -1$, alors $f * g$ existe au sens des distributions.

On peut se demander quel est dans ce cas l'ordre de $f * g$. Il n'existe peut-être pas des critères généraux à cet effet. Mais on peut trouver des réponses intéressantes dans des cas particuliers. Posons, par exemple,

$$h_y(x) = \frac{1}{x^2+y^2}\,,\ \ \forall\, y \in \mathbf{R}$$

Alors, on a la proposition:

Si f est une distribution telle que $f \in O(x^\alpha)$, où $\alpha < 1$, la convolution $h_y * f$ existe pour tout $y \neq 0$ et on a encore $h_y * f \in O(x^\alpha)$.

Observons que, si l'on pose dans ce cas

$$\varphi(x,y) = \frac{1}{\pi} h_y(x) * f(x) , \qquad \text{pour } x \in \mathbf{R}, y > 0 ,$$

φ est la solution de l'équation de Laplace $\Delta u = 0$ dans la demi-plan $y > 0$, telle que

$$\varphi(x, 0^+) = f(x) , \quad \varphi(x,y) \in o\left(\sqrt{x^2 + y^2}\right)$$

Ces résultats, que nous n'avons d'ailleurs pas eu l'occasion de vérifier en détail, sont en rapport avec l'étude de la transformation de Stieltjes (voir notre article «*La théorie des ultradistributions et les séries de multipôles des physiciens*», à paraître dans «Mathematischen Annalen»).

Nous estimons que ce point de vue pourra conduire à des résultats nouveaux, intéressants pour les physiciens, dans le cas de distributions de plusieurs variables.

Observons d'autre part que l'on obtient encore un critère probablement utile, en remplaçant dans la proposition 17.2 (corollaire de 17.1) la condition «$\alpha < -2$» par la condition «α entier $\leqslant -1$».