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On Automorphisms of Arbitrary Mathematical Systems

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Translator's summary
The translated paper is an extract, published in 1945, of an unpublished thesis, of both historical and technical import, dealing with notions of definability and their relation to invariance under automorphisms. The author develops a metamathematical Galois theory, and discusses and anticipates some aspects of higher-order model theory in an informal but conceptually rich manner.

Translator's introduction

The plan for this translation goes back to 1971, when I saw a reference to this paper by J. Sebastião e Silva (henceforth 'Silva') in Reyes *1970*. My incursion into logic was just beginning; I was then unaware of Silva's early interests in mathematical logic, but realized that this was a matter to be publicised and requiring some further study. The first draft of the translation was expected to be included in an edition of his works, but this project was held up for lack of financial support and other reasons. However, it was again taken up in 1982, on the occasion of the International Symposium and Colloquium on the tenth anniversary of his death, held in Lisbon in March 1982 and organized by the (reborn) Portuguese Mathematical Society.[1] This time, however, no translation is intended, but subsequent developments have pointed to the desirability of publishing this remarkable paper in English translation.[2]

A few words are in order on the life and career of José Sebastião e Silva. He was born on 12 December 1914, in the small town of Mertola, in the Alentejo province, of wheat plains, olive trees and cork-oaks, birthplace of many a renowned Portuguese mathematician. From the age of thirteen he was giving private lessons of Latin to his

1 The Proceedings of the Symposium on Functional Analysis and Differential Equations will be published in volume 41 of *Portugaliae mathematica*. The Proceedings of the Colloquium on the Teaching of Mathematics in the 1980s was published by the Portuguese Mathematical Society.

2 Volume II and III of the mathematical works of Silva from 1958 to 1972 have already been published under the title *Obras de José Sebastião e Silva* (Imprensa Nacional-Casa da Moeda, 1982–83). Volume I is in preparation and will include all of his earlier published and unpublished research work.

colleagues in the school at Beja, showing already great pedagogical talent which was later put to good use in admirable teaching of mathematics and in the writing of textbooks for the secondary level. In spite of ill health, he finished high school with the highest marks and went on, in 1937, to take a first degree in Mathematical Sciences, at the Faculty of Sciences at Lisbon. It seems that he would have preferred to be a physicist but, knowing the lack of adequate laboratories and research facilities, he took up mathematics instead, where he excelled. He read widely, not only up-to-date mathematics but also history and philosophy (against the prevailing trend in the academic world). His career was still to await five long and hard years of private tutorials and teaching in middle schools in the outskirts of Lisbon until, at the age of 28, he was called by Professor A.A. Monteiro to join the *Centro de Estudos Matemáticos de Lisboa*.[3] He advanced rapidly in his studies and research and published his first papers on algebra and topology in *Portugaliae mathematica*. He also began collaborating in *Gazeta de matematica* with a series of notes on the incorporation of logic notation in the teaching of mathematics at the elementary level (reviewed in the *Journal of symbolic logic*), and presented two communications to the Luso-Spanish Congresses of Mathematics in 1940 and 1942.

In 1942 Silva became second Assistant at the Faculty of Sciences, and a year later was awarded a scholarship in order to complete a dissertation for the Doctorate, at the *Instituto Nazionale di Alta Matematica*, Rome. These were difficult times from a political point of view, and this place was not of his choosing, right in the middle of the War. Nevertheless, he managed to survive the bombings and privations, produce valuable work and create friendships with the mathematical intelligentsia—Francesco Severi, Luigi Fantappié, Guido Castelnuovo and his daughter, Emma, Frederico Enriques and Mauro Picone, among others. It is conceivable that his interest in logical matters was enhanced by long discussions with Enriques at the home of the latter, whom he admired most for his conciliation of the formal aspects of logic with its intuitive basis in the historical background. It then comes as no surprise that his first dissertation dealt with this subject, broadly speaking, of which I shall say more later.

Silva's encounters with L. Fantappié were also to produce good fruit later on, after returning to Portugal in 1946. In 1949 he submitted a second thesis to the Faculty of Sciences, based on a new systematizatoin of Fantappié's theory of analytical functionals, and was awarded the Doctorate accordingly. This important work came to the notice of Gottfried Köthe in Heidelberg, who drew the mathematical world's attention to it, and was later taken up by Grothendiek and Tillmann.[4] But once more

3 This was a mathematical research center recently created under the State's main institution for the promotion of scientific research and the granting of scholarships in foreign countries: the *Instituto para a alta cultura*. Professor Monteiro, then Director of the *Centro*, was later expelled from the University, along with other mathematicians and scientists, on political grounds, by the dictator Salazar. He was a founding member of the old Portuguese Mathematical Society and of the journals *Portugaliae matematica* and *Gazeta de matemática*, and went on to found (in Brasil and Argentina) the school of mathematical logic in Latin America.

4 For details of Silva's important contributions to functional analysis, locally convex spaces, theory of distributions, etc. see the appraisal by Köthe *1973*.

the Faculty of Sciences closed its doors to the young Silva, who therefore applied successfully to a professorship at the Agronomical Institute, Lisbon, where he remained for nine years.

The peaceful surroundings, human and otherwise, gave Silva the most productive period of his life, at the end of which he was internationally acknowledged, elected member of the Academy of Sciences at Lisbon, and finally invited to become Professor of Mathematics at the Faculty of Sciences of Lisbon. He remained there till the end of his life in 1972, except for short semesters at the Faculty of Sciences of Oporto and at the University of Maryland, U.S.A., where he taught a course on his axiomatic approach to L. Schwartz's theory of distributions. During this period, he was also engaged in, and chiefly responsible for, the mathematical curricula reforms in the middle and high school levels, a project sponsored by the O.E.C.D., highly praised (but also largely misunderstood) and adopted in other countries. All told, the life and career of Silva's is worthy of a chapter of its own in Laurence Young's *Mathematicians and their times* (Young *1981*), as that of a great mathematician of our times. But let us now return to the lesser-known aspects of his work in logic, and to the subject matter of this translation.

During his stay in Italy, in 1943 and 1944, Silva wrote a 200-page thesis entitled 'Para uma teoria geral dos homomorfismos' ('Towards a general theory of homomorphisms'), which he intended to submit for the Doctorate degree on his return to Portugal. It seems that Enriques, after reading the thesis, advised him against this intention, for fear that it would not be understood or accepted at the time. This was probably good advice on those grounds, and in fact he never presented nor published the thesis in this form.[5] Instead, he published an extract containing results from the last chapter, which is the paper, Silva *1945*, now translated. It was presented to the Pontifical Academy by Academician Francesco Severi on 30 July 1945.

The paper begins with a description of logical and set-theoretic notation. Finitary predicates are identified with sets of *n-tuples* (from a fundamental set U), but the author also considers predicates of infinite degree and predicates of higher type (predicates of predicates, etc.), although the emphasis is on finitary predicates of type at most 2. A general theory of types over a set U is established by means of the power set operation. Many examples are presented, mainly from number systems and geometry.

The primitive logical operations are conjunction, negation, universal quantification, the description operator ι and replacement of (free) variables. Two concepts of logical definability or expressibility are introduced. The first of these (expressibility in the *normal* sense) corresponds to definability in the usual sense, i.e. in a finite number of steps, or by means of a finite formula constructed from primitives, logical and non-logical. The second concept, in the *wide* or larger sense, is such that 1) definability in the normal sense implies definability in the wide sense; 2) if every solution of a predicate α is definable in the wide sense from primitives then α too is so definable;

5 This thesis is presently being typeset and revised by me for publication, to be included in Volume I of *Obras* (footnote 2). It contains the proofs of the results stated in the translated paper.

3) definability in the wide sense is transitive.[6] The author points out that condition 2) is distinctive with respect to the first notion of definability, and asks whether the two notions are in fact equivalent. It seems to me that this second notion of definability is not as precisely stated as it might have been, or could be, for comparison purposes or otherwise. One way to look at it is as a form of *implicit* definability, but then a detailed analysis is only possible within a precise formal language, as done by Beth *1953* for first-order languages; but we may also look at it as involving possibly infinitary and other extensions of first-order languages, and this may well call for the techniques of abstract logic.[7]

The main body of the paper is devoted to mathematical systems (structures), automorphisms and their relation to definability. As said in the author's summary, the aim is to generalize to arbitrary mathematical systems the main propositions of Galois theory, and the terminology adopted (irreducible predicates, Galois group of a set, etc.) reflects this aim.[8] In describing this metamathematical Galois theory one must note that, for Silva, a mathematical system $[U; \mathfrak{P}]$, where U is a set and $\mathfrak{P}$ a list of primitives (predicates or relations, elements and operations on U) is defined up to interdefinability, in the sense that $[U; \mathfrak{P}_1] = [U; \mathfrak{P}_2]$ if and only if every primitive of one system is definable in terms of the primitives of the other. This is not the usual notion of structure in model theory, but is not so uncommon in mathematics, in topology say, where a topological structure can be given in a number of alternative interdefinable ways (family of open sets, closure operator, etc.). He may then state, as a consequence of the second fundamental theorem (section 18), that two systems are identical iff they have the same group of automorphisms. Another such consequence is that the logical closure of a subset A of the universe U of a system (i.e. the set of elements of U logically definable from the primitive predicates and parameters in U) is the set of elements of U which remain fixed by every automorphism of the system that leaves invariant the elements of A.

The fundamental theorems of sections 17 and 18 are the basis of his *metamathematical method*, which he applied successfully at least three times in his later work on functional analysis and the theory of distributions. The first instance of this application is in the aforementioned second thesis (*1950*, 40), the second in *1958*, 107, and the third in *1960*, 5. The details of these applications are too technical to be described here, but the general problem in the three cases was that of determining the general expression for the continuous linear mappings from one functional space into another.[9]

6 In his review of the paper, Bennett *1949* seems to me to have misunderstood the second concept of definability. For quantification, being a logical primitive, is already allowed in the first scheme of definability.

7 A clue to Silva's intended meaning is also to be found in his unpublished thesis, where he refers to explicit definition (normal sense) as a constructive one, and to definition in the wide sense as a descriptive one.

8 Similar programmes seem to have been devised by other people, such as Krasner *1938*.

9 I do not know of other references to the paper by Silva or the results therein other than the ones already given, but J. Corcoran has called my attention to the fact that 'definability implies invariance under automorphisms' (part of the second fundamental theorem, section 18) is discussed in Tarski and Lindenbaum *1934*, and was used in Tarski *1956* and Tarski and Beth *1956*. It underlies, of course, Padoa's method for the proof of undefinability.

Much more could be said about the content of this paper; some comments, by myself or by others, are incorporated into the editorial footnotes to the paper. Like editorial interpolations into the text, these footnotes are enclosed in square brackets,[10] in order to distinguish them from Silva's footnotes (to some of which my remarks have been added). All the footnotes have been incorporated into one scheme of numbering, which runs on from the one already in use; Silva's were numbered afresh from 1 on each page.

The original pagination of the paper is indicated *in situ* in square brackets. The original notations have been maintained (and, in places, explained in editorial footnotes); for clarity of reading a few long expressions have been displayed. Throughout, the style and terminology of the author has been maintained. The survey bibliography which he appended to the paper has been presented in the house-style of this journal; my own biblography comes at the end.

[327] ON AUTOMORPHISMS OF ARBITRARY MATHEMATICAL SYSTEMS

JOSE SEBASTIÃO E SILVA

SUMMARY. After defining general notions pertaining to mathematical systems and their automorphisms, the author obtains such conclusions that extend to the maximum doctrine expounded in Felix Klein's 'Erlangen Programme' in the field of general Galois theory. Moreover, these conclusions not only help organize and clarity many issues in all parts of mathematics, but can also prove useful in questions in functional analysis. The results stated herein shall be demonstrated in another work.

1. PREDICATES DEFINED ON A SET.—Given a set U with elements a, b, . . . of arbitrary nature, a *complex* of n elements of U is an arrangement of elements of U, possibly with repetitions, of length n. We say that a *predicate* α of *order* n, or n*-ary* is *defined on* U if there exists a criterion by means of which, given any complex of n elements of U, we can always assert whether or not the said complex *satisfies* the predicate α, and *one and only one of these two possibilities holds.*[11]

If a complex $(c_1, c_2, \ldots, c_n)$ satisfies a predicate α, we write $\alpha(c_1, c_2, \ldots, c_n)$. In case the predicate α is binary we may write $c_1 \; \alpha \; c_2$ instead of $\alpha(c_1, c_2)$. Needless to say that the meaning of $c_1 \; \alpha \; c_2$ need not coincide with that of $c_2 \; \alpha \; c_1$.

[328] If $x_1, x_2, \ldots, x_n$ denote *indeterminate* elements of U, then the formula $\alpha(x_1, x_2, \ldots, x_n)$ expresses a *conditional proposition* in $x_1, x_2, \ldots x_n$ *defined* in U, that is, a *variable* proposition which assumes one and only one of the truth values

10 Note, however, that the square brackets used in notations are Silva's own.

11 [*n-tuple* is the common name for a complex of n elements of U; however, the author later considers complexes of infinite length so, for the sake of uniformity, we maintain the author's terminology in this and most other respects. The author's wording of the notion of 'defined on U' is suggestive of some kind of effectiveness in the possibility of being able to decide, hence to assert that the predicate holds or does not hold of a given complex, rather than it just being the case that one of the alternatives holds (though we may not know which at present). Reading on, however, it seems that this last meaning is the intended one or that, in any case, it is the one that matters.]

'*true*', '*false*' whenever the variables $x_1, x_2, \ldots, x_n$ are replaced by adequate constants, that is, symbols representing certain elements of U.[12]

The *solutions* of an *n-ary* predicate α are the complexes of n elements of U which satisfy predicate α. Two predicates α, β are considered *identical*, written $\alpha = \beta$, if they admit the same solutions.

Predicates of order higher than one are called *relations*. Then the expressions '*satisfies the predicate* α' and '*satisfies the relation* α' are synonymous.

Example.—Representing by N the set of natural numbers [positive integers], let us write $p \dashv q$ as an abbreviation of '*p divides q*'. The formula $p \dashv q$ therefore expresses a condition in p, q defined on N, and the symbol $\dashv$ represents a binary relation defined on this set. Similarly the formula $p \equiv q$ (mod. m) with the usual meaning ('*p is congruent to q modulo m*') expresses a condition in p, q, m and, writing simply $\mu(p, q, m)$ for $p \equiv m$ (mod. m), the symbol μ thus represents a 3-ary relation on N. Finally, let us agree to write $\pi(n)$ with the meaning '*n is even*', and thus obtain an example of a unary predicate on N, but if we represent by P the *set* (or *class*) of even numbers, the same condition can be expressed in the form $n \varepsilon$P, where the symbol ε stands, as usual, for the expression: '*is an element of*', or '*belongs to*'.[13]

2. LOGICAL OPERATIONS.—The conditional propositions defined on a given set U can be combined with each other by means of [329] *logical operations*, thus producing new conditional propositions still defined on U, as follows:

1) The *conjunction* (or *logical product*) of two conditional propositions **p**, **q** is the proposition **p** ∧ **q** (read '**p** and **q**') which is satisfied for those and only those assignments to the variables that satisfy *simultaneously* the propositions **p**, **q**.

2) The *disjunction* (or *logical sum*) of two conditional propositions **p**, **q** is the proposition **p** ∨ **q** (read '**p** or **q**') which is satisfied for those assignments that satisfy at least one of the propositions **p**, **q**.

3) The *negation* of a proposition **p** is the proposition ∼**p** (read 'not **p**') which is satisfied by those and only those assignments to the variables that *do not* satisfy **p**.

These three operations are not independent of each other, since to assert **p** ∨ **q** is equivalent to asserting ∼(∼**p**∧∼**q**), whatever **p**, **q** may be (first DE MORGAN's law). On the other hand, it is convenient to introduce the derived logical operator → (*material implication*) by convening to write **p** → **q** with the meaning accorded to **q**∨∼**p**; the symbol → thus replaces the locution '*if*... *then*'.

It is easy to see that the propositions **p** ∧ **q**, **p** ∨ **q**, **p** →**q** remain conditional in the same variables as **p**, **q** put together; similarly for ∼**p**. There exist, however, logical operators (the *quantifiers*) which reduce the number of [free] variables in the propositions to which they apply: in order to express that a given proposition $\alpha(x)$,

12 [The term 'propositional function' would also be adequate and even natural, since the author's notation and terminology owes much to Russell's, and even to Peano's. Note, however, that the author does not define a precise formal language, by modern standards, but seems rather to work in some higher order language of types with principal interpretations only, or even in the language of set theory (see section 8).]

13 One of the main advances of mathematical Logic over Aristotelian Logic consists in the fact that in the latter a simple or atomic judgement is always composed of a single subject and a unary predicate, whereas in the former a simple judgement may contain several subjects and a predicate of larger order.

conditional in a unique variable x, and defined in U, is satisfied by *all the possible values* of x (elements of U), we shall write $\bigcap_x \alpha(x)$; in order to express that *there exists at least one value* of x which renders $\alpha(x)$ true, we write $\bigcup_x \alpha(x)$. The symbols $\bigcap_x$, $\bigcup_x$ will therefore replace the expressions: '*For whatever x* . . . ' and '*There exists at least one x such that*' respectively, and we see how such operators (called *quantifiers*), when applies to conditional propositions in a single variable, originate *categorical propositions*, that is propositions with a well determined value ('*true*' or '*false*'). In fact, the truth of the propositions $\bigcap_x \alpha(x)$, $\bigcup_x \alpha(x)$ no longer depends on the variable x, which, for this reason, becomes an *apparent* variable, comparable to a dummy index under a summation sign.

However, the quantifiers can also be applied to propositions conditional in several variables, and in this case it is clear that a quantifier, [330] by rendering *apparent* one variable and leaving *free* the remaining variables, produces yet a conditional proposition. In any case, the formation of categorical propositions of a theory is not possible without direct or indirect recourse to the quantifiers, applied iteratively until all variables are rendered apparent. For example, the formula

$$\bigcup_p \bigcap_n [(n = p) \vee (n > p)]$$

expresses a proposition in the theory of natural numbers.

Finally, let us note that the quantifiers are not independent from one another, for instead of $\bigcup_x$ **p** we may write $\sim \bigcap_x \sim$ **p**, for any proposition **p** (second DE MORGAN's Law).

3. SIMPLIFYING THE NOTATION.—In order to alleviate the symbolic expressions and thus bring the language of mathematical Logic closer to ordinary language it is convenient to introduce the following conventions:

1) We may write $\bigcap_{x,y}$ for $\bigcap_x \bigcap_y$, $\bigcup_{x,y,z}$ for $\bigcup_x \bigcup_y \bigcup_z$, etc.

2) Instead of $\bigcap_{x,y,\ldots}$ $(\mathbf{p} \rightarrow \mathbf{q})$ we simply write $\mathbf{p} \xrightarrow[x,y,\ldots]{} \mathbf{q}$, and $(\mathbf{p} \xrightarrow[x,y,\ldots]{} \mathbf{q}) \wedge (\mathbf{q} \xrightarrow[x,y,\ldots]{} \mathbf{p})$ is abbreviated to $\mathbf{p} \overset{\longleftarrow}{\underset{x,y,\ldots}{\longrightarrow}} \mathbf{q}$.

3) If the variables rendered apparent by the symbol $\xrightarrow[x,y,\ldots]{}$ include all the variables free in **p**, **q** then $\mathbf{p} \xrightarrow[x,y,\ldots]{} \mathbf{q}$ is a categorical proposition, which may be written $\mathbf{p} \subset \mathbf{q}$. The symbol $\subset$ (*formal implication*) thus replaces the expression '*if* . . . *then, necessarily*'; on the other hand, the proposition is commonly expressed by saying that '**p** *is a sufficient condition for* **q**', or that '**q** *is a necessary condition for* **p**'.

4) Whenever $\mathbf{p} \subset \mathbf{q}$ and $\mathbf{q} \subset \mathbf{p}$ both hold we may simply write $\mathbf{p} \equiv \mathbf{q}$ and say that the propositions **p**, **q** are *formally equivalent*.[14]

4. CHANGING OF VARIABLES.—Amongst the fundamental logical operations must be included the *changing of variables*. Let us take an example: 1) represent by U the class of human beings, and [331] write $x \varphi y$ with the meaning '*x is a son or daughter*

14 [*The reason for the author's use of* '$\subset$' for formal implication, where other authors continue to use the same sign as for material implication ('$\supset$', '$\rightarrow$' or even '$\xrightarrow[x\ldots]{}$') becomes apparent in section 6, where $\subset$ is related to set-theoretical inclusion. This can be said also of the author's preference of $\bigcap_x$, $\bigcup_x$ for the quantifiers.]

of y'. Then the symbol φ will represent a binary relation, defined on U. If we now adopt the abbreviation $x\,\theta\,y$ for '*x is an uncle or aunt of y*', then:

$$(x\,\theta\,y) \equiv \bigcup_{u,\,v} [(x\,\varphi\,u) \wedge (v\,\varphi\,u) \wedge (y\,\varphi\,v)].^{15}$$

Thus we see that the *new* concept θ is defined in terms of the concept φ by means of: 1) two substitutions of variables which transform the conditional proposition $x\,\varphi\,u$ into its *conjugates* $v\,\varphi\,u$, $y\,\varphi v$; 2) two conjunctions; 3) a double quantification.

We have therefore an example of a *logical definition* which is not possible without the changing of variables. This same example shows how a given predicate φ can be realized into infinitely many conditional propositions $x\,\varphi\,y$, $x\,\varphi\,z, \ldots$ conjugates of each other (conjugates, but not equivalent!).

For linguistic convenience we sometimes say '*the predicate* $\alpha(x, y, \ldots)$' instead of '*the predicate* α'.

5. EFFICIENT PREDICATES.—Let us start again with a concrete example: write $\theta(x, y)$ with the meaning

$$\frac{2x}{y^2 + 1} > 0,$$

where x and y are real variables. We then have that $\theta(x, y) \equiv (2x > 0)$, which shows that the binary relation θ is *immediately* expressible in a unary predicate.

We shall say that a predicate α of order n is *efficient* it if cannot be immediately expressed in terms of predicates of lesser order.[16]

6. PREDICATES AND THEIR SETS OF COMPLEXES.—Given n sets $A_1, A_2, \ldots, A_n$, the (*cartesian*) *product* of these sets (in the order by which they are written) is the set denoted $A_1 \odot A_2 \odot \ldots \odot A_n$ whose elements are the complexes $(a_1, a_2, \ldots, a_n)$ with $a_1, a_2, \ldots, a_n$ representing elements of $A_1, A_2, \ldots, A_n$ respectively. [332] If all these sets coincide, say

$$A_1 = A_2 = \ldots = A_n = \mathrm{U},$$

the product of these sets may be represented by $\mathrm{U}^{(n)}$.

Let A be a subset of $\mathrm{U}^{(n)}$. We call *projection of A* along the n-th *coordinate* the set of complexes $(a_1, a_2, \ldots, a_{n-1})$ obtained by suppressing the last element a_n in some complex $(a_1, a_2, \ldots, a_n)$ of A

This much said, note that to each n-ary predicate, α, defined on U there

15 [Alan Slomson remarked that this definition is not correct unless one adds the conjunct $\sim(v = x)$.]

16 [The author used 'effective' (italian *effectivi*), which we have changed to 'efficient', not only because of the usual logical meaning of the term 'effective' but also because the author's other use of the idea of effectiveness in section 10 does not seem consistent with the notion defined in this section. John Corcoran points out that the concept of 'expressed *immediately*' is not mathematically precise (it tacitly presupposes rules of inference which are never mentioned), but nothing in the paper depends on how this is defined exactly. This type of omission can perhaps be traced back to Peano's style in his *1889*.]

corresponds a subset of $U^{(n)}$: the set of solutions of α; and, conversely, to every subset A of $U^{(n)}$ there corresponds the n-ary predicate:

$$(x_1, x_2, \ldots, x_n) \varepsilon A.$$

Moreover, given two n-ary predicates α, β defined on U, if we represent by A, B the sets of solutions of α, β respectively we find that in the above correspondence:

1) The *conjunction* $\alpha(x_1, \ldots, x_n) \wedge \beta(x_1, \ldots, x_n)$ is translated into the *intersection* $(A) \cap (B)$ (and, similarly, the *disjunction* into *reunion*).

2) The *negation* $\sim \alpha(x_1, \ldots, x_n)$ corresponds to the formation of the *complement* $\sim(A)$.

3) The quantification $\underset{x_n}{\cup} \alpha(x_1, \ldots, x_n)$, with $n > 1$, is translated into the projection along the n-th coordinate.

(Thus, for example, if x, y are real variables, the conditional proposition $\underset{z}{\cup} (x^2 + y^2 + z^2 = 1)$, equivalent to $x^2 + y^2 \leq 1$, represents a disc in analytical geometry, which is the projection on the (x, y)-plane of the spherical surface of equation $x^2 + y^2 + z^2 = 1$.[17])

4) The *changing of variables* corresponds to *changing the order* [of the coordinates].

5) The *unconditional implication* $\alpha(x_1, \ldots, x_n) \subset \beta(x_1, \ldots, x_n)$ corresponds to the *inclusion* $(A) \subset (B)$, and the *equivalence* $\alpha(x_1, \ldots, x_n) \equiv \beta(x_1, \ldots, x_n)$ to the *identity* $(A) = (B)$.

From these facts it follows that from the point of view of pure Logic the n-ary predicates defined on U can be conceptualized as subsets of $U^{(n)}$. In this manner Logic becomes identified with the theory of sets in the form of a developed combinatorial Analysis.[18]

[333] 7. FORMAL EXPLICATION: OPERATIONS AND FUNCTIONS.—Given a unary predicate, α, which admits one and only one solution, we shall denote by $\iota_x \alpha(x)$ this solution. Thus, the symbol ι_x (which renders apparent the variable x) stands for the expression '*the element x such that . . .* '. But the said symbol can also be applied to propositions with several variables. For example, let again U be the class of human beings and adopt the formulas $x \varphi y$, $\mu(x)$ as abbreviations of '*x is son or daughter of y*', '*x is male*' respectively; then the formula $\iota_y((x \varphi y) \wedge \mu(y))$ represents not a well-determined element of U but rather a variable, y, *function* of the independent

17 [Leon Henkin noted that the concept of cylindrification, rather than that of projection, is the natural way to provide a geometric meaning for the logical operation of existential quantification. Thus the disc in question would be represented by the pair of relations $x^2 + y^2 \leq 1$, $z = 0$, since the first of these represents a solid cylinder in 3-dimensional analytic geometry.]

18 If we searched for the reason which has led men to distinguish between a predicate and the set of objects (complexes) for which the predicate holds we would fall outside the scope of this work. However, once a fundamental set U is fixed, the definition is only nominal. [The identification proposed in the last sentence of the text is perhaps too strong a claim to take literally, but is acceptable if one restricts to the so-called algebra of classes (or of sets) and relations over a given domain, in the sense of the usual Boolean operations, together with projections; or rather with cylindrifications, by the previous note.]

variable x. If we write *Pad* x as an abbreviation of $\iota_y((x\,\varphi\, y) \wedge \mu(y))$ then we say that the constant *Pad* represents a unary operation defined on U.

More generally, let $\alpha(x_1, \ldots, x_n, y)$ be a relation of order $n + 1$ defined on U such that for every complex $(x_1, \ldots, x_n)$ in a certain subset (C) of $U^{(n)}$, there exists one and one only element y of U that satisfies the condition $\alpha(x_1, \ldots, x_n, y)$. Thus, introducing a new constant φ by means of the equivalence

$$[y = \varphi(x_1, \ldots, x_n)] \equiv \alpha(x_1, \ldots, x_n, y) \wedge (x_1, \ldots, x_n) \varepsilon \mathrm{C}$$

we see that φ represents a rule that assigns, to each complex in C, a certain element of U. We then say that φ represents an *operator* (or an *operation*) of *order n*, or *n-ary, defined on* (C), and that y is a *univocal function* of $x_1, \ldots\ x_n$ defined on (C). We therefore see how the concepts of operation and of function are contained in the concept of relation. An operation of order n is none other than a particular relation of order $n + 1$ put in *operational* or *explicit* form.[19]

8. THE THEORY OF TYPES.—Consider a fundamental set U. Other than the subsets A, B, ... of U we may consider sets $\mathfrak{A}$, $\mathfrak{B}$ ... of subsets of U, sets of sets of subsets of U, etc. Let us represent by Cls. U the set of all subsets of U (including U itself and the empty set O). The symbol Cls. may, naturally, be applied several times in succession. In general, the elements of Cls. U shall be called sets of *type* 1 with respect to U: [334] the elements of Cls.Cls. U sets of *type* 2 with respect to U, and so on. This formation of ever new entities can be pursued still through the transfinite: representing by Cls.$^\omega$ U the set of all sets of *finite* type, that is the union

$$\text{Cls. U} \cup \text{Cls.Cls. U} \cup \text{Cls.Cls.Cls. U} \cup \ldots,$$

we find in Cls.$^\omega$ U an example of a set of type ω with respect to U; by continuing the application of the symbol Cls. and taking unions at appropriate stages we obtain sets of type $\omega + 1, \omega + 2, \ldots, 2\omega, \ldots,$ with respect to U.[20]

But, taking in account what was said in [section] 6, all such sets correspond to unary predicates. It is easy to see, though, how to apply the hierarchy of types to predicates of arbitrary order. For let $\Gamma(\xi, \eta, \ldots; x, y, \ldots)$ be a conditional proposition in variables $\xi, \eta, \ldots$ for predicates defined on U (variables of type 1); and, possibly, variables $x, y, \ldots$ for elements of U (fundamental variables, or variables of type O). We then say that the constant Γ represents a predicate of *type* 2 over U. Similarly we define predicates of types 3, 4, ... over U, and continue this way through the transfinite. On the other hand, the elements of U can be considered as *predicates* of type 0.

Example: In the set N of natural numbers, let $\Phi(\zeta)$ abbreviate the condition

19 [The author is probably aware that in ordinary logic and set theory one can do without the description operator ι as a basic logical operation, but little use is made of it anyway. Its inclusion is another indication of the author's affiliation to Russell.]

20 [It is clear that the author is thinking of *cumulative* types. On the other hand, predicates of arbitrary type are allowed into other predicates, so there seems to be no question of separation or restriction of types as in Russell's theory. Silva seems to have had a dislike for the axiomatic foundations of set theory—see the next footnote.]

$$[\zeta(x, y) \underset{x,y}{\longrightarrow} \zeta(x, y + x)] \wedge \underset{x}{\cap} \zeta(x, x).$$

It is clear that Φ represents a unary predicate of type 2 over N, which suits infinite binary relations ζ, of type 1 (for instance, the relations $x \leq y$, $x \dashv y$, etc.). If, however, we consider the predicate Θ defined by

$$\Theta(\zeta) \equiv [\zeta(x, y) \underset{x,y}{\rightleftarrows} \zeta(x, y + x)] \wedge [\zeta(x, y) \underset{x,y}{\longrightarrow} x \leq y] \wedge \underset{x}{\cap} \zeta(x, x)$$

we see that Θ admits one and only one solution:—the relation 'x is a factor of y' or, briefly, $x \dashv y$. We may therefore write

$$[\iota_\zeta \, \Theta(\zeta)] \, (x, y) \equiv (x \dashv y).$$

Note also that the classification according to type is applicable to operators since, as we have seen, these are particular forms of relations. Thus, for instance, *the operations of derivation and integration are of type 2 with respect to the set of real numbers (or the set of complex numbers); functional equations and, in particular,* [335] *differential equations are simply conditional propositions of type 2 over the same set*, etc., etc.

Finally, the logical constants ($=$, $\wedge$, $\vee$, $\sim$, etc.) can be realized as relations and operations of different types, according to the types of the relations to which they are applied. For example, the conjunction $\wedge$, when applies to relations of type 1, acts as an operation of type 2, etc.

It is illicit to speak of the *totality* of relations over a given set U, otherwise one falls into RUSSELL's famous paradox.[21]

9. THE THEORY OF TYPES AND LOGICAL SYMBOLISM.—When considering predicates of finite type we can always tell their type just by looking at the arrangement of parenthesis comprised in their full expressions. Thus, for example, a relation α [whose expression is] of the form

$$\alpha_{z,\eta}(x, y, \xi_{u,v}(z, \eta(u, v)))$$

shall be of type 3, if x, y, u v represent variables of fundamental type and ξ, η variables of higher type (bounding those variables indicated as indexed); this relation α corresponds, following an interpretation analogous to that indicated in [section] 6, to a subset of the cartesian product

$$\mathrm{U}^{(2)} \odot \mathrm{Cls.}(\mathrm{U} \odot \mathrm{Cls.}\, \mathrm{U}^{(2)}),$$

where U represents the fundamental set.

21 [Of course we can speak indirectly of the said totality even if U is a *proper class*, and directly if U is a *set*, in the usual sense of these terms in axiomatic set theory. The remark by the author only makes sense in the context of naive set theory (Cantor), which contemplates no such distinctions. It is symptomatic that the author's bibliography includes no reference on axiomatizations of set theory proper.]

If the relations are presented as operations, the writing conventions are naturally more complicated, but the problem can still be completely solved for operators of finite type.

However, when trying to extend to the transfinite the logical symbolism, interesting difficulties arise, which bring to light the very foundations of our reasoning. In fact, the study of transfinite numbers shows that, although it is always possible to expand a given formalism with new conventions, it is no longer possible to establish a rigorous formalism which comprises, definitively, all types corresponding to ordinal numbers of class II. This fact, directly related to the well-known paradox of RICHARD, constitutes the main difficulty in the theories of definition and of proofs, and in particular explains the failure in realizing LEIBNIZ's dream relative to a universal scientific language.[22]

[336] 10. THE TWO CONCEPTS OF LOGICAL DEFINITION.—Let us consider as *fundamental* the operations of: *conjunction* ($\wedge$), *negation* ($\sim$), *quantification* ($\underset{x}{\cap}$), *formal explication* (ι_x) and *changing of variables*.

Let us also suppose given predicates α_1, α_2 of any type, over U, which we call *primitive predicates*, and let α be another predicate over U.[23] We say that α is *logically expressible, in the normal sense*, in the primitive predicates $\alpha_1, \alpha_2, \ldots,$ whenever α can be construed as the last term of a *finite* sequence, whose first terms are primitive predicates, possibly followed by logical relations and variables $\xi_1, \ldots, \xi_p$ of types higher than 0, and whose remaining terms are predicates obtained from preceeding terms in the sequence by a single application of a fundamental logical operation.[24] The categorical proposition, in the form of an identity or a formal equivalence, which establishes the meaning of the new symbol α as a logical function of the symbols α_1, $\alpha_2, \ldots$ will be called, naturally, the *logical definition* of α.

Thus, for example, returning to the examples in [section] 4, the concept '*uncle* (or *aunt*) *of*' is logically expressible, in the normal sense, in the concept '*son* (or *daughter*) *of*'; and, as seen in [section] 8, the concept '*divides*' is logically expressible, in the normal sense, in the concepts + and <.

To the expression *logically expressible in the wide sense* we associate the more restricted meaning compatible with the following conditions: 1) If the predicate α is

22 [In earlier work on ordinals and order types, limit ordinals are referred to as *ordinal numbers of the second kind* or *class* (i.e. *class* II), successor ordinals and zero being of the *first class* (or *class* I). This is not to be confused with the *second number-class* which contains exactly all infinite countable ordinals, the *first number-class* being the class of finite ordinals. If the author's reference to ordinals of class II is taken as referring to ordinals of the second number-class, then we may suppose that he is stating that Hilbert's formalisms (formal languages with decidable syntax, etc.) do not extend readily to uncountable languages. Even so, or otherwise, the connection with Richard's paradox (having to do with different meanings of the term 'definable', the point being that some such meanings cannot be formalized simply) remains obscure to the translator.]

23 Not excluding, naturally, the possibility of having elements $a, b, \ldots$ of U as primitive (predicates of type 0) which, for convenience, should be rendered in the form of unary predicates of type 1: $x = a, x = b, \ldots$

24 For the sake of simplicity and better understanding we may suppose in addition that all primitive predicates and the variables introduced are of finite type. This will have no appreciable effect in the results to be considered.

logically expressible in the predicates $\alpha_1, \alpha_2, \ldots$, *in the normal sense*, then it is also expressible in the *wide sense*; 2) If *every* solution of α is logically expressible in the wide sense in the predicates $\alpha_1, \alpha_2, \ldots$ then α itself is logically expressible, in the wide sense, in the predicates $\alpha_1, \alpha_2, \ldots$; 3) If α is logically expressible, in the wide sense, in the predicates $\beta_1, \beta_2, \ldots$ and each of these is in turn logically expressible, in the wide sense, in the predicate $\alpha_1, \alpha_2, \ldots$, then so is α.

The difference between [337] the former and the latter concepts lies manifestly in the second condition.[25] Let α, for example, be a unary predicate of type 1 whose solutions $a, b, \ldots$ are all logically expressible, in the normal sense, in the primitive predicates; then we shall have, as logical definition of α:

$$\alpha(x) \equiv (x = a) \vee (x = b) \vee \ldots.$$

If the solutions $a, b, \ldots$ are finite in number, it is clear that in this definition we do not go beyond the scheme of definition in the normal sense. But can the same be said in general? More precisely: can it be asserted that every predicate α, logically expressible in the *wide sense* in the primitive predicates, is also expressible in the *normal sense*? This is a very delicate problem, of the nature of problems relative to ZERMELO's [Choice] Principle, to the non-contradiction of Arithmetic, to the question of solvability of any given mathematical problem, etc., which also relates closely to the difficulties mentioned in the preceding section.[26]

If α has infinitely many solutions, the term 'expressible' refers only, naturally, to a purely *ideal* possibility of definition, because of the limitation of our mind in following an infinite number of independent operations; instead of '*logically expressible in the primitive predicates*' it would perhaps be preferable, for example, to use an expression like '*logically determined with respect to the primitive predicates*'. It is not said, however, that the above question ought to be answered in the negative, *at least in the case in which every individual element is logically expressible, in the normal sense, in the primitive predicates*. One may ask, for instance, in the case of integers, if it is permitted to assert the existence of sets or sequences which do not conform to any finitistic law of formation; that is, which preclude any possibility of *effective* definition. As is well known [338], this question has aroused no little discussion among mathematicians of different tendencies.[27]

But let it be said that there should not be excluded altogether the possibility of a predicate α logically expressible in the normal sense in certain predicates $\alpha_1, \alpha_2, \ldots$

25 In other words, this condition merely expresses the fact that a predicate is univocally determined by its solutions.

26 For example, it is easily proved that, in any given class of ordinal numbers, every element is logically expressible, in the wide sense, in the primitive relation $<$; on the other hand, it does not seem easy to prove that every ordinal number is logically expressible, in the normal sense, in the relation $<$.

27 The above concept of '*logical expressibility in the normal sense*' corresponds fairly closely to the borelian concept of '*definable with a finite number of words*', amplified with LEBESGUE's '*nameable*'. It should be noted that BROUWER's intuitionistic school admits only a very strict notion of logical definition which, however, comprises recurrent definitions, as was deeply studied by the Hilbertian school (see the [author's] bibliography).

being also expressible in the wide sense, but not in the normal sense, in certain other predicates $\beta_1, \beta_2, \ldots$; and it is this very doubt that makes the second of the two concepts introduced really interesting.

11. PREDICATES OF INFINITE ORDER.—In mathematical language there are frequently present predicates of infinite order. Thus, for example, if we convey to write $\gamma(\alpha_1, \alpha_2, \ldots, a_n, \ldots)$ to mean that the denumerable sequence of real numbers $(\alpha_1, \alpha_2, \ldots, a_n, \ldots)$ *converges, we say that the constant* γ represents *a predicate of order* ω, *defined on the set* R *of real numbers*. Similarly, the symbol $\lim\limits_{n \to \infty}$, with its usual meaning, represents *an operator of order* ω, *defined on the subset of* $\mathrm{R}^{(\omega)}$, consisting of all convergent sequences.

Although predicates of infinite order can always be reduced to unary predicates of higher type, it may be more *convenient*, in some arguments, to retain the former.

In the sequel we denote a finite or infinite complex $(a_1, a_2, \ldots)$ by $\bar{a}_i$, where the index i represents a variable over a given class of ordinal numbers or, more generally, over an arbitrary set $\mathbf{R}$, called the *set of positions of the complex* $\bar{a}_i$. On the other hand, the notation $\{a_i\}$ is reserved for the *set* of the elements a_i, abstracting from the order or repetition of the said elements in the complex $\bar{a}_i$.

[339] 12. THE CONCEPT OF MATHEMATICAL SYSTEM.—Given a set U, and supposed fixed a set $\mathfrak{P}$ of predicates $\alpha_1, \alpha_2, \ldots$ over U, we say that the set U, together with the predicates $\alpha_1, \alpha_2, \ldots$ as primitives, constitutes a *mathematical system* $[\mathrm{U}, \alpha_1, \alpha_2, \ldots]$, and in addition we say that two mathematical systems $[\mathrm{U}, \alpha_1, \alpha_2, \ldots]$ and $[\mathrm{U}, \beta_1, \beta_2, \ldots]$ are *identical*, in symbols

$$[\mathrm{U}, \alpha_1, \alpha_2, \ldots] = [\mathrm{U}, \beta_1, \beta_2, \ldots],$$

if and only if the following two conditions are satisfied: 1) the predicates $\alpha_1, \alpha_2, \ldots$ are logically expressible (possibly in the wide sense) in the predicates $\beta_1, \beta_2, \ldots$; 2) the predicates $\beta_1, \beta_2, \ldots$ are logically expressible (possibly in the wide sense) in the predicates $\alpha_1, \alpha_2, \ldots$. The primitive predicates can also be presented in the form of operators or sets. Instead of the notation $[\mathrm{U}, \alpha_1, \alpha_2, \ldots]$ we can also use this other, more condensed: $[\mathrm{U}; \mathfrak{P}]$, but then one must not lose sight of the fact that, according to our previous conventions, the two systems $[\mathrm{U}; \mathfrak{P}]$, $[\mathrm{U}, \mathfrak{P}]$ may be distinct.[28]

Instead of '*mathematical system*' we may simply say '*system*', whenever there is no danger of confusion.

Examples.—Let us represent by P the set of points of ordinary space, regarded as ordered triples of real points, and agree to write $\mathrm{Rt}(x, y, z)$, $\mathrm{Tr}(x, y, z)$, $\mathrm{Eg}(x, y, u, v)$ with the meanings '*x, y, z are co-linear*', '*y is between x and z*', '*the distance from x to y is equal to the distance from u to v*' respectively, where x, y, u, v are variables over

28 [The author's definition of *identical* systems (Italian *identici*) is a major departure from standard practice and should be borne in mind throughout the rest of the paper. The basic idea is that of equivalence of presentation with respect to definability (in either sense), and the terminology chosen should have been more suggestive in this respect. Nonetheless we have maintained the original terminology in this printed version of the translation (the first version, circulated for some time, contained some major alterations in notation and terminology), following a suggestion by the Editor. The only difference is that we put a bold $\boldsymbol{=}$ instead of the ordinary equality sign $=$ used by the author.]

P.[29] Let us agree further to write $u = \mathrm{Sm}(s, y, z)$ in order to express the same fact that, in vectorial language, is translated by the formula

$$u = z + (x - z) + (y - z);$$

and write $y = \lim_n x_n$ with the usual meaning. Then it is easy to see that

$$[\mathrm{P}, \mathrm{Rt}, \mathrm{Tr}] = [\mathrm{P}, \mathrm{Sm}, \lim_n], \text{ but } [\mathrm{P}, \mathrm{Rt}, \mathrm{Tr}] \neq [\mathrm{P}, \mathrm{Rt}, \mathrm{Tr}, \mathrm{Eg}],$$

since the predicate Eg is not logically expressible, not even in the wide sense, in the predicates Rt, Tr.[30] Note finally that the systems [P, Rt, Tr], [P, Rt, Tr, Eg] are none other than *ordinary affine space* and *ordinary Euclidean space* respectively.—

Although no restriction has been imposed on the orders or types of the primitive predicates $\mathfrak{P}$, the fact is that, in practice, very seldom do we encounter systems in which the primitive predicates [340] are of type higher than 2 or of order larger than ω. On the other hand, as we shall see later, *any system is definable by means of a finite number of predicates of finite order and type not higher than* 2. The topological systems are characterized by a primitive predicate of type 2, usually in the form $x \varepsilon X'$ ('*x is an accumulation point of X*') *which, in all usual spaces, except the real line, is unreplaceable by predicates* of lower type. In the algebraic systems, on the contrary, the primitive predicates are generally of type 1 and presented in the form of operations.[31]

13. THE CONCEPT OF AUTOMORPHISM.—Given a fundament set U, let θ be a *biunivocal transformation* of the set U on itself, that is an operator which, to each element of U, associates an element $\theta(x)$ of U in such a way that, given any element y of U, there always exists one and one only element x of U such that $y = \theta(x)$. Then, given a complex $(a_1, a_2, \ldots)$ of elements of U, we shall call the *image* of this complex, by means of θ, the complex $(\theta(a_1), \theta(a_2), \ldots)$; given a predicate α defined in U, we call the *image of* α *by* θ, and represent by $\theta(\alpha)$, the predicate whose solutions are the images by θ of the solutions of α; and finally we say that θ *preserves* (or *leaves invariant*) the predicate α if, and only if, the condition

$$\alpha(x_1, \ldots, x_n) \equiv \alpha(\theta(x_1), \ldots, \theta(x_n))$$

is satisfied, that is, the condition $\theta(\alpha) = \alpha$ holds.

Let then α be a predicate of type 2 over U. As we know, the solutions of α are complexes consisting of predicates defined on U and, possibly, elements of U. We still call *image of* α *by means of* θ, and represent by $\theta(\alpha)$, the predicate whose solutions are the

29 HILBERT uses Gr (from *Gerade*) instead of Rt, Zw (from *zwischen*) instead of Tr.

30 We also have that [P, Rt, Tr] = [P, Tr], and [P, Tr, Eg] = [P, Eg].

31 Note that groups, fields, etc. correspond to a stricter notion of mathematical system than that adopted here. A similar distinction is to be found between the notions of '*straight line*' and '*oriented line*'. For these systems the expression '*qualified system*' may be introduced. [Thus a qualified system is just a system satisfying some 'axioms' expressing properties of its relations, etc.]

images by θ of the solutions of α; and we say that the transformation θ *preserves* the predicate α [341] if, and only if, $\theta(\alpha) = \alpha$. Similarly for predicates of higher type.

This much said, we call *automorphism* of a system $[U; \mathfrak{P}]$ any biunivocal transformation of the set U on to itself which leaves invariant the primitive predicates $\mathfrak{P}$.

We see immediately that the *product* of two automorphisms is still an automorphism, and how the *inverse* transformation of an automorphism is an automorphism still; this means that *the set of automorphisms of a given mathematical system forms a group, with respect to the usual product of transformations.*

Examples.—The automorphisms of a topological space are the *bi-continuous biunivocal transformations of the space* in itself. The automorphisms of the system [P, Rt, Tr] are the *spatial affinities.* The automorphisms of the system [P, Rt, Tr, Eg] are the *spatial similarities*. The group of similarities is a proper subgroups of the group of affinities. In general, we shall see below, *the adjunction of a primitive predicate reduces or conserves the group of automorphisms, according as to whether it alters or not the given system.*[32]

14. INDISCERNIBLE ELEMENTS. DEFORMABLE SYSTEMS.—A mathematical system is said to be *deformable* or *undeformable* according as to whether it admits or not admit an automorphism other than the identity. For example, the system [N, suc] where N represents the set of natural numbers and suc the unary operation '*successor of*', is an undeformable system, and the same can be said of the systems $[N, <]$, $[N, +]$ since

$$[N, \mathrm{suc}] = [N, <] = [N, +];$$

but not as much can be said of the system $[N, \times]$ which is certainly deformable. Analogously, the systems $[R, <]$, $[R, +, <]$, $[R, \times, <]$, where R represents the set of real numbers, are deformable; but the system $[R, +, \times, <]$ is undeformable.[33]

A sufficient condition for a system to be undeformable is that every of its elements be logically expressible (in either sense) in the primitive predicates. Is this condition also necessary? We have [342] here a problem, which presents itself as a particular case of this other problem, to be studied below: In any system, all predicates logically expressible in the primitive predicates remain invariant under every automorphism of the system; we then question: Is the converse also true?

32 [Presumably, alterations in the sense of $=$; see footnote 28.]

33 Note that $[R, +, 1, <] = [R, +, \times, <]$ but not $[R, +, 1] = [R, +, \times]$. However, $[R, +, \times] = [R, +, \times, <]$ since

$$(x < y) \equiv \bigcap_z (z \times z = y - x).$$

[The undeformabilities of the systems [N, suc], $[N, <]$ and $[N, +]$ is easily established directly, whereas $[N, \times]$ is deformable by any bijection that permutes two primes, say, and leaves the other primes fixed. Note also that multiplication, as a ternary relation, is not even (first-order) definable in the system $[N^*, 0, \mathrm{suc}, +, <]$ where N^* is the set of natural numbers, including zero. The function $x \mapsto cx$ (c positive real) deforms $[R, <]$ and $[R, +, <]$, and the function $x \mapsto x^3$ deforms $[R, \times, <]$. As for the system $[R, +, \times, <]$, a bijection which preserves $+, \times, <$ must be the identity on the rationals and cannot be discontinuous at any point. Likewise for $[R, +, 1, <]$. See also sections 17 and 18.]

With respect to such problems it is interesting to bring into evidence certain concepts, [as follows]. In a given mathematical system, two predicates α, β (of the same type and order) are said to be *discernible* if, and only if, there exists at least one unary predicate Γ logically expressible in the primitive predicates which is satisfied by one of the predicates α, β but not by the other.[34] In a given mathematical system, a predicate is said to be *isolable* if, and only if, it is discernible by means of the primitive predicates, from every other predicate over the same fundamental set. To the two concepts of '*logical expressibility*' there correspond, naturally, two concepts of '*discernible predicates*' and two others of '*isolable predicate*'. On the other hand, these concepts can be applied, in particular, to the elements of the fundamental set (predicates of type 0); and can be further extended to complexes of predicates.

As we shall see ahead, the existence of at least two predicates α, β distinct but indiscernible by means of the primitive predicates, is equivalent to the existence of at least one automorphism (different from the identity) which transforms one into the other; equivalent, therefore, to the deformability of the system. This fact is not equivalent, however, to the permutability of the two predicates. In fact, two predicates α, β are said to be *permutable* or *ambiguous* in a given mathematical system if, and only if, there is at least one automorphism θ such that $\theta(\alpha) = \beta$, $\theta(\beta) = \alpha$; and a necessary and sufficient condition for the two predicates to be permutable is that the complexes (α, β), (β, α) be indiscernible in the given system. For example, in the system [R, <], two distinct elements are indiscernible, but never permutable; in ordinary euclidean Geometry, that is in the system [P, Tr, Eg], two similar sets are always indiscernible, but only two sets symmetrical with respect to a point, a straight line or a plane are permutable;[35] in the field of complex numbers, every pair [343] of indiscernible elements (complex conjugates) is a pair of ambiguous elements; in the field Ra($\sqrt{5}$) there are, on the other hand, certain pairs of indiscernible elements which are not ambiguous, etc., etc.

15. TWO KINDS OF ISOMORPHISMS BETWEEN SUBSETS OF A SYSTEM.—Given a fundamental set U, let C be a subset of U and α a predicate over U. In addition, let α^* represent the predicate whose solutions are: 1) the solutions of α formed solely by elements of C—if α is of type 1; 2) those solutions of α formed solely by predicates defined in C and, possibly, elements of C—if α is of type 2; and so on, to the transfinite. We then say that α^* is none other than the predicate α, *relativized* to C.

This much said, consider a system [U; $\mathfrak{P}$]. Given two subsets C_1, C_2 of U and a biunivocal transformation θ of C_1 into C_2, we say that:

34 [Recall the definition of a set X of elements *indiscernible* in a structure (A, . . .) with respect to a linear order < on A: for any n and all finite sequences $x_1 < \ldots < x_n$, $y_1 < \ldots < y_n$ of elements of X ∪ A, the structures $(A, \ldots, x_1, \ldots, x_n)$ and $(A, \ldots, y_1, \ldots, y_n)$ are elementary equivalent, i.e. satisfy the same sentences in the appropriate first-order language (Chang and Keisler *1977*, 147). The basic idea of this definition is clearly a particular case of the author's notion of (a pair of) indiscernible predicates, preceeding Ehrenfeucht and Mostowski *1956* by more than ten years. Other concepts of a model-theoretic character in this and the next section deserve also a closer look in the light of modern developments. See also footnote 43.]

35 Needless to recall that the geometry of ordinary physical space is, more precisely, the euclidean metric Geometry, characterized by the group of congruences. In this case, the indiscernible figures are just the congruent ones. We shall not insist any further on this point.

a) θ is an *hypoisomorphism* of C_1 into C_2 if, and only if, it preserves every primitive predicate, $\mathfrak{P}$, relativized to C_1.

b) θ is an *hyperisomorphism* of C_1 into C_2 if, and only if, it preserves every predicate logically expressible in the primitive predicates $\mathfrak{P}$ and relativized to C_1.

The two sets C_1, C_2 are said to be *hypoisomorphic* (*hyperisomorphic*) whenever there exists at least one biunivocal transformation of C_1 into C_2 which is an hypoisomorphism (hyperisomorphism).

For example, in the topological Cartesian plane, a parabola is hyperisomorphic to a straight line; but a segment without its extremities is not hyperisomorphic to a straight line, though it is hypoisomorphic to it (or homeomorphic, as usually said in this case). As we shall see, the concept of '*hyperisomorphic sets*' coincides with that of '*indiscernible sets*'.

16. LOGICAL BASIS. LOGICALLY CLOSED SETS. IRREDUCIBLE PREDICATES.—Given a system [U; $\mathfrak{P}$], let E be a subset of U and α a predicate over U (possibly of type 0). Then: *a*) we say that α is *determined in* E (with respect to the given system) whenever it is logically [344] expressible in the predicates $\mathfrak{P}$, possibly expanded with elements of E; *b*) we shall call set *logically generated from the elements of* E, or *logical closure* of E (in the system [U; $\mathfrak{P}$]), the set F of all elements of U which are determined in E; *c*) we shall call *logically closed* in the system given those sets that coincide with their logical closure; *d*) and call *logical basis* of E (in the given system still) any subset of E in which every element of E is determined. More generally, we call *logical basis* of a set E *with respect to* a subset A of E, or, briefly, *logical basis* of E/A, any set that is a logical basis of E, whenever the primitive predicates are expanded with the elements of A. We may further consider logical basis formed by predicates: this concept is defined in a similar manner as above.

Correspondingly to the two concepts of '*logical expressibility*' there turn out two concepts of '*predicate determined in a set*', likewise two of '*logical closure*', etc.

Examples.—In the geometrical spaces, either affine or euclidean, the logically closed sets are the linear varieties; in the fields of algebraic numbers, the logically closed sets are the subfields, etc., etc. It is to be noted that, in general, the empty set is not logically closed in the algebraic systems, but is so in the topological and geometrical systems.—

We can now introduce one of the most important concepts in this matter, which generalizes the known concept of '*irreducible equation in a given field*'. Let us take the system [U; $\mathfrak{P}$], and consider again: E, a subset of U; α, a predicate over U. We say that the predicate α is *irreducible* in the set E whenever the following two conditions are satisfied: 1) α is determined in E; 2) there is no predicate β, determined in E, which admits as solutions a proper part of the solutions of α.

Let now $(\alpha_1, \alpha_2, \dots)$ be a complex of predicates over U, and agree to write $\Gamma(\xi_1, \xi_2 \dots)$ in order to indicate that '*the complex* $(\xi_1, \xi_2 \dots)$ *is indiscernible, in the wide sense, from the complex* $(\alpha_1, \alpha_2, \dots)$ *by means of the predicates* $\mathfrak{P}$, *expanded with the elements of* E'. We may then realize that the predicate Γ is not only determined in E (at least in the wide sense), but is *precisely that predicate irreducible in* E *which admits* [345] *the complex* $(\alpha_1, \alpha_2, \dots)$ *as solution.* In simple words: *The predicate* Γ

represents the maximum that can be said of that complex when, to that effect, only the concepts $\mathfrak{P}$, *expanded with the elements of* E, *are employed.*

For example in affine Geometry, the irreducible predicate which admits a given circle as solution is the *genre ellipse*; in euclidean Geometry, on the other hand, the irreducible predicate which admits the same solution is the *species circle* (both predicates being of type 2 over P).[36]

17. THE FIRST FUNDAMENTAL THEOREM. IRREDUCIBILITY CRITERIA.—Consider the system [U; $\mathfrak{P}$] yet, and let: A be a subset of U; $\{a_i\}$ a logical basis of U/A. This basis may be finite or infinite and, for convenience, we suppose it orderly disposed in a complex $\bar{a}_i$. Let also ϱ be the predicate irreducible in A which admits the complex $\bar{a}_i$ as solution. We then have that:

The automorphisms of the system [U; $\mathfrak{P}$] *which leave fixed the elements of* A *are all the biunivocal transformations* θ *of the set* U *in itself which are obtained by replacing the complex* $\bar{a}_i$ *by an arbitrary solution* $\bar{b}_i$ *of the predicate* ϱ, *and associating every element* x *of* U *with that element* $\theta(x)$ *of* U *which is expressed in* $\bar{b}_i$ *in the same way as* x *is expressed in* $\bar{a}_i$.[37]

In each concrete case we naturally seek, for the irreducible predicate ϱ, a definition in the normal sense, or, better still, a constructive one, which is not claimed to be always possible. From the two fundamental theorems [see the next section for the second of these] the following *general irreducibility criteria* can be derived, easily applicable in practice in most cases:

Suppose known a *biunivocal* (or *canonical*) representation of the elements of U in the logical basis $\{a_i\}$; that is, that is known a family $\mathfrak{R}$ of univocal operators η, determined in A [336] and put in biunivocal correspondence with the elements of U, in such a way that, given any element x of U, we shall have $x = \eta(\bar{a}_i)$, with η the operator corresponding to x.[38] Then, *a necessary and sufficient condition for* η *to be irreducible in* A *is that, for any predicate* α *of type 1, primitive or of the form* $x = a$ *with* $a \varepsilon A$, *the predicate* Γ *such that*

$$\bigcup_{\bar{x}_i} [\alpha(\eta_1(\bar{x}_i), \eta_2(\bar{x}_i), \ldots) \wedge \varrho(\bar{x}_i)] \equiv \Gamma(\eta_1, \eta_2, \ldots)$$

be determined in A, *that is, independent of the basis* $\{a_i\}$. And similarly for the possibly primitive predicates of higher type.[39]

Examples.—One of the most important examples that can be mentioned in this context, but which I abstain from developing, concerns the algebraic extension of fields. By way of illustration we present solely the following two cases:

a) Consider the affine space [P, Rt, Tr], and let A be a subset of P formed by two

36 It is clear that the concept of 'irreducible predicate' generalizes that of 'shape configuration' [Italian '*corpo di figure*'] which was introduced in the classification of geometries by means of the theory of groups.
[By a slip, Silva numbered the next section '16'.]

37 Although seeming almost evident, the proof of this theorem is anything but immediate. Note, however, that in it the principle of ZERMELO does not intervene.

38 Such a representation always exists, at least in the zermelian sense.

39 The primitive logical predicates [operations?] are supposed included.

points, p, q. The logical closure of A, in the wide sense, is the straight line $\overline{pq}$. Let us write sec(x, y, u, v) as an abbreviation of

$$\bigcup_z [\mathrm{Rt}(x, y, z) \wedge \mathrm{Rt}(u, v, z)];$$

a minimal logical basis in the wide sense of P/A is formed by two points a, b such that

$$\sim \sec(p, q, a, b) \wedge \sim \sec(p, a, q, b) \wedge \sim \sec(p, b, q, a),$$

that is, such that they are not *coplanar* with the points p, q. Then, if we agree to indicate this condition by the formula $\varrho(a, b)$, we find ϱ to be the irreducible predicate in A which admits (a, b) as solution. Any element x in P can be put in the form $x = \varphi(a, b)$ where φ is an operator resulting from a finite or infinite superposition of the operators Sm, lim ([section] 12), and substitutions of elements of A for variables; therefore, an operator determined in A. We thus obtain a biunivocal representation of the elements of P. And from this it is easily seen how to determine the automorphism of the space which leaves the elements p, q fixed.

[347] If we now join the primitive concepts Rt, Tr with the concept of congruence Eg, thus turning to Euclidean Geometry, the condition for the pair (a, b) to form a logical basis of P/A is still the same, but the predicate irreducible in A which admits (a, b) as solution becomes restricted, and no longer symmetrical, in general. The automorphisms of the system which leave fixed the elements p, q are the rotations about the line $\overline{pq}$ and the symmetries with respect to the planes which contain $\overline{pq}$.

Another interesting example to examine is the one corresponding to the previous two, in projective Geometry.[40]

b) Take the system $[\mathrm{N}^*, \times]$ where N* represents the set of natural numbers adjoined with 0. The logical closure of the empty set, in this system, is formed by the elements 0, 1. On the other hand, it is easy to see that the prime numbers constitute a minimal logical basis, albeit an infinite one, of the given system.[41] Every [positive] number is uniquely representable by means of its decomposition into prime factors. Then, if we write $\pi(p_1, p_2, \ldots)$ to indicate that '*$p_1, p_2, \ldots$ are the prime numbers, without omission or repetition*', we easily see that π is the irreducible predicate (in the empty set) which admits the complex $(2, 3, 5, \ldots)$ as solution. The automorphisms of the given system shall then be the transformations θ which, to every number a of the form $p_1^{n_1} \times \ldots \times p_k^{n_k}$, where $p_1, \ldots, p_k$ represent prime numbers, associates the number

$$\theta(a) = [\theta(p_1)]^{n_1} \times \ldots \times [\theta(p_k)]^{n_k},$$

where $(\theta(2), \theta(3), \ldots)$ is any permutation of the sequence of prime numbers.

Another very interesting and instructive example is that of the system $[\mathrm{R}, \times, <]$. A logical basis of this system can be formed with any real number different from 0, 1 and -1. The automorphisms are all the transformations θ of the form

40 In general, all the results obtained by KLEIN and POINCARÉ in the systematization of Geometry using the concept of group can be cited as examples.

41 Another system which admits an infinite minimal logical basis is Hilbert space. On the other hand, the field of algebraic numbers admits no minimal basis. Finally, the systems $[\mathrm{R}, +]$, $[\mathrm{R}, \times]$ admit minimal basis with the power of the continuum, accepting ZERMELO's principle.

$$\theta(x) \equiv \frac{x}{|x|} \cdot |x|^c$$

with c real and non zero.—

[348] Note in addition that *the first fundamental theorem extends readily to the case of logical basis formed by predicates*, and only in this general form does it intervene in establishing ulterior results. On the other hand, as we shall see, there is always the possibility of finding a finite logical basis formed by predicates of type 1.

Example.—Let us take again the system [R, ×, <]. A logical basis for this system can be taken as formed by the operator +, submitted to the following properties:

$(x + y) + z \equiv x + (y + z)$ $\qquad$ (4) $x < y . \subset . x + u < y + u$

(2) $x + y \equiv y + x$ $\qquad$ (5) $\bigcap_{x,z} \bigcup_{y} (x + y = z)$

$$(3)\ x \times (y + z) \equiv x \times y + x \times z.$$

But these properties represent the *maximum* that can be said of addition, using only the concepts of × and <; that is, if we put an operator variable φ in place of +, and represent by $\Gamma(\varphi)$ the conjunction of said properties, we find in Γ the irreducible predicate which admits the solution +. The solutions of Γ, that is the operations which are *binary, univocal, associative, commutative, distributive with respect to multiplication, monotone and invertible*, shall then be all the operations φ given by the expression

$$x \varphi y \equiv \theta(\theta^{-1}(s) + \theta^{-1}(y)),$$

where θ represents an operator of the form indicated in the preceding example (that is, an automorphism). We see, however, that the concept of addition is not a very *convenient* logical basis for the determination of the automorphisms of the system [P, ×, <]; all the same, an interesting problem was solved.

Equally suggestive is the corresponding case in the system [N*, ×] where the infinite logical basis {2, 3, 5, . . . } can be replaced by the finite basis {+}. These two examples suffice to give an idea of how fruitful the point of view introduced can become in functional Analysis, specially in view of the progress already achieved in the field of logical integration of differential equations.

[349] 18. THE SECOND FUNDAMENTAL THEOREM.—This theorem can be stated in the following manner: *Given a system* [U; $\mathfrak{P}$] *and a predicate* α *over* U, *a necessary and sufficient condition for* α *to be logically expressible, in the wide sense, in the predicates* $\mathfrak{P}$, *is that it remain invariant for all the automorphisms of the system.*[42]

The proof is based on the following assumption: Given a system [U; $\mathfrak{P}$], it is always possible to determine a logical basis of U formed by a *finite* number of predicates of

42 This theorem is admitted *intuitively*, at least, for particular kinds of systems (geometrical, topological, etc.), but no doubt the queston deserves a rigorous analysis.

put in biunivocal correspondence with sets constructed, in last analysis, from the integers or, more generally, from a class of ordinal numbers; and, accordingly, there always exist, naturally, predicates of type 1, finite in number, which render logically individualized, at least in the wide sense, all the elements of the set.[43]

In particular, we may say that, if the predicate α, which remains invariant for all automorphisms of the system considered, is logically expressible, in the normal sense, in the basic predicates $\beta_1, \ldots, \beta_n$ and if, in addition, the irreducible predicate which admits the complex $(\beta_1, \ldots, \beta_n)$ as solution is logically expressible, in the normal sense, in the predicates $\mathfrak{P}$, then the same can be said of the predicate α.

[Two] immediate consequences of this theorem: I) *Every mathematical system is univocally determined by its group of automorphisms; that is, two systems are identical if, and only if, their respective automorphism groups coincide.*[44] II) *In any system* [U; $\mathfrak{P}$], *the logical closure of a subset* A *of* U *is formed by those elements of* U *which remain fixed by every automorphism of the system which leaves fixed the elements of* A.

The second of these corollaries generalizes a well known proposition of GALOIS theory. The converse of this reads: To every [350] subgroup $\mathfrak{H}$ of the group of automorphisms there corresponds a set V logically closed in the system, in such a way that the group of automorphisms which leave fixed the elements of V is precisely the group $\mathfrak{H}$. This, however, is the only fundamental proposition of GALOIS theory which fails to generalize here. We shall see in the last section the proposition which replaces this one in the genreal case.

19. DETERMINATION OF THE HYPERISOMORPHISMS OF A SET. GALOIS GROUP OF A NORMAL SET.—Given a system [U; $\mathfrak{P}$], let A be a subset of U, B a subset of A, and $\{a_i\}$ a logical basis of A/B. Let also ϱ be the predicate irreducible in B which admits the solution $\bar{a}_i$. Then it is easily seen that the hyperisomorphisms of A which leave fixed the elements of B can be obtained in a manner similar to that which was indicated in the first fundamental theorem, from the automorphisms of the system. If, in particular, A is logically closed in the system, and all the [elements of the] solutions of ϱ belong to A, we can further say that the images of A, by means of those hyperisomorphisms, all coincide with A; and in this case this set will be said to be *normal with respect to* B, in the given system. It is then natural to call the group of hyperisomorphisms of A which leave fixed the elements of B the *Galois group* of A/B. From this definition it follows that, if A is normal with respect to B and α is a predicate irreducible in B which admits a solution in A, then every solution of α is likewise in A.

We can also prove that: *Every hyperisomorphism between two subsets of a*

43 [The exact status of the assumption on which the proof of the second fundamental theorem is said to be based should perhaps be looked into closely. The lowering of the types of the predicates of a logical basis is suggestive of a kind of reducibility axiom, as in the ramified theory of types. Their finiteness amounts to the fact that, in any stage of development of a (formal) theory one can only have a finite definitional extension of the primitive language. But, of course, this could be infinite to start with (i.e. with infinitely many non-logical symbols), as we now know. On the other hand, the last part of the paragraph seems to presuppose some sort of 'constructibility' ('ordinal definability'?) of the sets of the universe in the sense of Gödel.]

44 [This passage should be read with footnote 28 in mind.]

system is extendable to an automorphism of the system. Hence: *The pairs of hyper-isomorphic sets in a system are none other than the pairs of indiscernible sets in the system.*

Furthermore, it can be proved that: *Given a system* [U; $\mathfrak{P}$], *a subset* A *of* U *and a subset* B *of* A, *if* θ *represents one of the transformations in the Galois group of* U/B, *and* $\mathfrak{H}$ *the Galois group of* U/A, *then* $\theta \mathfrak{H} \theta^{-1}$ *is the Galois group of* U *with respect to* θ(A). Hence: *A necessary and sufficient condition for a subset* A *of* U *to be normal with respect to its subset* B *is that the Galois group of* U/A *be an invariant* [or *normal*] *subgroup of the Galois group of* U/B. [351] In addition: *Given a system* [U; $\mathfrak{P}$] *and three subsets of* U, C ⊂ B ⊂ A, *such that* A *is normal with respect to* B *and* B *is normal with respect to* C, and representing by $\mathfrak{G}$, $\mathfrak{G}'$, $\mathfrak{H}$ *the Galois groups of* A/C, A/B, B/C *respectively, then* $\mathfrak{H} \cong \mathfrak{G}/\mathfrak{G}'$.

These results are further examined from the points of view of the two concepts of logical definition, and can be extended still by means of an adequate concept of normal families of predicates.

20. GALOIS GROUP OF A PREDICATE. DECOMPOSITION OF A PREDICATE INTO A LOGICAL SUM OF IRREDUCIBLE PREDICATES.—Given a system [U; $\mathfrak{P}$] and a subset A of U, let α be a predicate of type 1, determined in A, and let V be the logical closure of the set A, amplified with all the elements that compose the solutions of α. It then becomes easy to see that V is normal with respect to A; and in these conditions it is natural to say that the Galois group of V/A is also the *Galois group* of the predicate α with respect to A, which can thus be conceived as a group of transformations of the solutions of α.

Then the following generalizations of a known result of Galois theory can be established: *Given a system* [U; $\mathfrak{P}$] *and a subset* A *of* U, *every predicate of type* 1, *determined in* A, is univocally decomposable into a disjunction of predicates irreducible in A, *each of which admits as solutions the elements of a transitivity cell determined over the set of solutions of* α, *by its Galois group with respect to* A.

21. ORGANIZATION OF A SYSTEM DERIVED FROM THE GROUP OF AUTOMORPHISMS. AUTONOMOUS SYSTEMS AND WIENER SYSTEMS.—Consider the following problem: *Given a group* $\mathfrak{G}$ *of biunivocal transformations of a set* U *in itself, to determine a family* $\mathfrak{P}$ *of predicates over* U *in such a way that the group of automorphisms of* [U; $\mathfrak{P}$] *be exactly* $\mathfrak{G}$.[45] This problem is always soluble, once admitted the hypothesis [352] formulated in [section] 18. In effect, let $\{\beta_1, \ldots, \beta_n\}$ be a finite family of predicates of type 1, in terms of which all elements of U are logically expressible, possibly in the wide sense, and let us represent by Γ the predicate (of type 2) which has as solutions all complexes which are the transforms of $(\beta_1, \ldots, \beta_n)$ by means of the transformations in the group $\mathfrak{G}$; then, the system sought is clearly the system [U, Γ]. It should be noted that the problem is uniquely determined in the sense that a given group cannot give rise to several distinct systems.[46] It is however convenient to try to reduce the primitive predicates to a minimal number of predicates of minimal type and order. But there are cases in which it is not possible to obtain reduced predicates

45 A particular instance of this is the problem of determining the geometry, given a Lie group.

46 [Again, footnote 28 is relevant.]

of type 1: this happens, for example, with cartesian topological spaces of several dimensions, the group of automorphisms of which are *ω-transitive*, that is, such that, given to complexes of *n* fundamental elements, there always exists an automorphism which transforms one into the other (them being indiscernible, therefore), but this fact is not satisfied for any two denumerable complexes. *All the same, it is a relevant fact that any system is definable by a finite number of predicates of type not higher than 2.*

In particular, given a system [U; $\mathfrak{P}$] and a subgroup $\mathfrak{H}$ of its automorphism group, there always exists a family of predicates of type not higher than 2 which, when added to the primitive predicates $\mathfrak{P}'$, reduce to $\mathfrak{H}$ the group of automorphisms of the system. This is the very proposition which replaces that other proposition of Galois theory that was mentioned at the end of [section] 18.

In trying to solve the previous problem in a more restricted mode, we obtain the following result: Given a group $\mathfrak{G}$ of biunivocal transformations of a set U in itself, a necessary and sufficient condition for $\mathfrak{G}$ to be the group of automorphisms of the system [U, $\mathfrak{G}$] is that it is not an invariant subgroup of any group distinct from $\mathfrak{G}$. Let us call *autonomous* those systems that satisfy this condition. An important family of autonomous systems is that family of topological systems which we name after WIENER: *a Wiener system* is a mathematical system [U; $\mathfrak{P}$] which coincides with the topological system [U, $\mathfrak{F}$], $\mathfrak{F}$ representing the family of sets logically closed in the system [U; $\mathfrak{P}$]. To determine a necessary and sufficient condition on the group $\mathfrak{G}$ for the system [U, $\mathfrak{G}$] to be a Wiener system is the main issue underlying the Wiener problem, [353] in a new formulation.[47] A necessary condition is that one regarding the autonomous systems, which was already found, by other route, in a work of mine on the problem of Wiener.[48] But such a condition is not sufficient.

Examples.—Every geometric affine space is a WIENER system, in which the family of logically closed sets is formed by the linear varieties. The Euclidean spaces, on the other hand, are autonomous systems which are not WIENER systems, since their family of logically closed sets is still formed by the linear varieties. Finally, the Euclidean metric spaces are not even autonomous systems, since the group of congruences is always an invariant subgroup of the group of similarities.

[355]

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