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III.1

THEORY OF DISTRIBUTIONS*

* Este texto tem por base apontamentos coligidos por diversos alunos de José Sebastião e Silva na sequência de um curso que realizou em 1958 na Universidade de Maryland, e que posteriormente foram utilizados, e por ele revistos, na Faculdade de Ciências de Lisboa.

CHAPTER VI

LIMITS AND INTEGRALS OF DISTRIBUTIONS

6.1. Limits of a distribution as $x \to +\infty$

Let I be an open interval unbounded on the right; i.e. of the form $I =]a, +\infty[$ with $a \in /R \cup \{-\infty\}$. The following two definitions are well known in classical analysis.

6.1.1. DEFINITIONS. Let f and φ be two functions on I. The function f is said to be **of order less than** φ iff $\exists\ x_0 \in /R$ and a function f_0 such that:

$$f = \varphi f_0 \quad \text{for} \quad x > x_0 \quad \text{and} \quad f_0(x) \to 0 \quad \text{as} \quad x \to +\infty.$$

On the other hand, f is said to be **at most of the order of** φ as $x \to +\infty$ iff $\exists\ x_0 \in /R$ and a function f_0 bounded for $x > x_0$, such that $f = \varphi f_0$.

In the first case we shall write:

$$f \in o(\varphi) \text{ as } x \to +\infty \text{ (or, on the right)}$$

and in the second case:

$$f \in O(\varphi) \text{ as } x \to +\infty \text{ (or, on the right).}$$

These relations replace the classical $f=o(\varphi)$ and $f=O(\varphi)$ which are not logically correct and may produce confusion in functional analysis.

Observe that:

6.1.2. *If there exists x_0 such that $\varphi(x)\neq 0$ for $x>x_0$, then*

$$f\in o(\varphi) \text{ as } x\to+\infty \Leftrightarrow \frac{f(x)}{\varphi(x)}\to 0, \text{ as } x\to+\infty$$

$$f\in O(\varphi) \text{ as } x\to+\infty \Leftrightarrow \frac{f(x)}{\varphi(x)} \text{ is bounded on the right.}$$

In order to extend "*o*" to distributions, we first consider the case where $\varphi=\hat{x}^{\alpha}$, with $\alpha>-1$ (for simplicity the sign "^" will be omitted).

Let be $\Im$ the Lebesgue integral operator defined by $\int_c^x f(\xi)d\xi$ with c in I.

6.1.3. LEMMA. *If α is a real number >-1 and f a continuous function such that $f\in o(x^{\alpha})$ as $x\to+\infty$, then $\Im f\in o(x^{\alpha+1})$ as $x\to+\infty$.*

PROOF. Suppose $f\in o(x^{\alpha})$ as $x\to+\infty$. This means that there exists x_0 and f_0 such that $f=x^{\alpha}f_0$ for $x>x_0$ and $f_0\to 0$ as $x\to+\infty$. Let $\delta>0$ be given; then $\exists x_1\in I\!R$ such that $|f_0(x)|<\delta$ for all $x>x_1$. We can assume $x_1>x_0>0$. Now, for every $x>x_1$

$$\Im f(x)=K+\int_{x_1}^{x}\xi^{\alpha}f_0(\xi)d\xi \quad \text{where} \quad K=\int_c^{x_1}f.$$

Since $|f_0(x)|<\delta$ and $\xi>0$, for $\xi>x_1$, we have:

$$\left|\frac{\Im f(x)}{x^{\alpha+1}}\right|\leq\frac{|K|}{x^{\alpha+1}}+\frac{x^{\alpha+1}-x_1^{\alpha+1}}{(\alpha+1)x^{\alpha+1}}\,\delta,\quad \forall x>x_1,$$

and therefore, since $\alpha>-1$:

$$\lim_{x \to \infty} \left| \frac{\Im f(x)}{x^{\alpha+1}} \right| \leq \frac{\delta}{\alpha+1} .$$

As δ is arbitrary, this implies that

$$\frac{\Im f(x)}{x^{\alpha+1}} \to 0 \text{ as } x \to +\infty \text{; i.e. } \Im f \in o(x^{\alpha+1}). \ \blacklozenge$$

6.1.4. Remark. This lemma obviously extends to locally summable functions and even to measures, as we shall see.

The lemma suggests the following:

6.1.5. DEFINITION. Let α be a real number > -1 and f a distribution on I. We write $f \in o(x^{\alpha})$ as $x \to +\infty$ iff there exists an integer $p \geq 0$ and a continuous function F on I, such that:

$$f = D^{p}F \text{ and } \frac{F(x)}{x^{\alpha+p}} \to 0 \text{ as } x \to +\infty .$$

6.1.6. Remark. The lemma implies that if there exists $p \in IN_0$ and $F \in C(I)$ satisfying the preceding conditions, *then every integer* $m \geq p$ *and every function* G *such that* $G = \Im^{m-p}F + P$ *where* $P \in \mathcal{P}_m$, *satisfies the same conditions* (observe that if $P \in \mathcal{P}_m$, then $\dfrac{P(x)}{x^{\alpha+m}} \to 0$ as $x \to +\infty$).

6.1.7. LINEARITY PROPERTY. *If* $f \in o(x^{\alpha})$ *and* $g \in o(x^{\alpha})$ *as* $x \to +\infty$, *with* $\alpha > -1$, *then:*

$$\lambda f + \mu g \in o(x^{\alpha}) \text{ as } x \to +\infty, \ \forall \lambda, \mu \in \mathbb{C}.$$

For the proof, it is sufficient to represent f and g as derivatives of the same order of continuous functions, taking into account 6.1.6.

In particular, α *may be equal to* 0. Then $x^0=1$ and if $f\in o(1)$ as $x\to+\infty$, it is natural to say that $f\to 0$ as $x\to+\infty$. More generally, let λ be any complex number and $f\in\mathscr{D}(I)$; then:

6.1.8. DEFINITION. We say that f **converges to** λ as $x\to+\infty$ if and only if $f-\lambda\in o(1)$ as $x\to+\infty$. A distribution f is said to be **convergent** as $x\to+\infty$ if and only if $\exists\lambda\in\mathbb{C}$ such that $f\to\lambda$, as $x\to+\infty$.

Taking definition 6.1.5. into account and observing that $\lambda=D^p\left(\dfrac{\lambda x^p}{p!}\right)$ for every $p\in I\!N_0$, we can define the preceding concept as follows:

6.1.9. DEFINITION. We say that $f\to\lambda$ as $x\to+\infty$ if and only if there exists $p\in I\!N_0$ and $F\in C(I)$ such that:

$$f=D^pF \text{ and } \frac{F(x)}{x^p}\to\frac{\lambda}{p!} \text{ as } x\to+\infty \text{ (in the ordinary sense).}$$

Remark. Instead of "f tends to λ as $x\to+\infty$", we shall sometimes write "$f(x)\to\lambda$, as $x\to+\infty$", but it should be remembered that in these cases x is a dummy variable.

6.1.10. *If $f\to\lambda$ as $x\to+\infty$ and $f\to\mu$ as $x\to+\infty$ then $\lambda=\mu$.*

In fact, if $f-\lambda\to 0$ and $f-\mu\to 0$ as $x\to+\infty$, then, by 6.1.7. $(f-\lambda)-(f-\mu)=\mu-\lambda\to 0$ as $x\to+\infty$. But, for every integer $p\geq 0$ and every continuous function F such that $\mu-\lambda=D^pF$, we have necessarily $F=(\mu-\lambda)\dfrac{x^p}{p!}+P$ where $P\in\mathscr{P}_p$.

Hence by definition 6.1.9., $\mu-\lambda$ cannot tend to 0 *unless* $\lambda=\mu$.

This makes legitimate the definition complementary to 6.1.8.

6.1.11 DEFINITION. We say that λ is the **limit** of f as $x \to +\infty$, iff $f \to \lambda$ as $x \to +\infty$. In this case, we shall write $\lambda = \lim_{x \to +\infty} f(x)$ or $\lambda = f(+\infty)$.

The uniqueness of the limit is guaranteed in 6.1.10, and from 6.1.7., follows:

6.1.12. LINEARITY PROPERTY. *If f and g are convergent as $x \to +\infty$, then:*

$$\lim_{x \to +\infty} (\alpha f + \beta g) = \alpha \lim_{x \to +\infty} f + \beta \lim_{x \to +\infty} g, \quad \forall \alpha, \beta \in \mathbb{C}.$$

In turn, from 6.1.3. and the preceding definitions, it follows:

6.1.13. *If f is a continuous function such that $\lim_{x \to +\infty} f(x) = \lambda$ in the ordinary sense, then the same fact holds in the distributional sense; i.e., in the sense of definitions* 6.1.11. *and* 6.1.9..

Observe, that according to 6.1.5., this theorem extends to locally summable functions (and even to measures). However, it must be observed that the converse of this theorem is not true.

6.1.14. Example. As is well-known, the function $\cos x$ is not convergent in the ordinary sense as $x \to +\infty$. But we have:

$$\lim_{x \to +\infty} \cos x = 0, \text{ in the distributional sense.}$$

To see that, it is enough to apply definition 6.1.5. observing that $\cos x = D \sin x$ and $\dfrac{\sin x}{x} \to 0$, as $x \to +\infty$.

6.1.15. General remark. All preceding definitions may be extended and all propositions remain true, if we replace throughout $+\infty$ by $-\infty$ and "on the right" by "on the left". In particular, we must then consider an interval I, unbounded on the left, $I =]-\infty, a[$, instead of an interval unbounded on the right.

6.1.16. DEFINITION. We say that f **tends to** λ as $x \to \infty$ and we write $\lim\limits_{x\to\infty} f(x)=\lambda$ if and only if $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to-\infty} f(x)=\lambda$.

For example, it is easily seen that (cf. 6.1.14): $\lim\limits_{x\to\infty} \cos x=0$ (in the distributional sense).

6.2. Limits and value of a distribution at a point of $\mathbb{R}$

Let now I be any open interval $]a, b[$, *bounded on the left*. Then, definitions 6.1.1. and 6.1.2. are readily extended to this case, replacing throughout "$x\to+\infty$" by "$x\to a^+$" and "on the right" by "on the left."

If we place $\Im_a f(x)=\int_a^x f(\xi)d\xi$, we prove, as for 6.1.3. (the proof is even simpler):

6.2.1. LEMMA. *If f is a continuous function on I, such that $f\in o[(x-a)^\beta]$ as $x\to a^+$ where $\beta>-1$,* then $\Im_a^n f\in o[(x-a)^{\beta+n}]$ *as $x\to a^+$,* for $n=0, 1, \ldots$.

This lemma justifies the following

6.2.2. DEFINITION. If $f\in\mathscr{D}'(I)$ and $\beta>-1$, we write $f\in o[(x-a)^\beta]$ as $x\to a^+$ iff there exists $p\in\mathbb{N}_0$ and $F\in C(I)$ such that $f=D^pF$ and

$$\frac{F(x)}{(x-a)^{\beta+p}}\to 0 \text{ as } x\to a^+.$$

6.2.3. Remark. The lemma implies that if there exist p and F satisfying these conditions, *then every integer $m\geq p$, along with the function $\Im_a^{m-p}F$, satisfies the same conditions*. (But it must be observed that for each integer $m\geq p$, there is no function different from $\Im_a^{m-p}F$ satisfying the same conditions).

Now we are able to extend definitions 6.1.8. and 6.1.11., as well as propositions 6.1.7., 6.1.10., 6.1.12. and 6.1.13., replacing $+\infty$ by a^+. In particular, the convergence as $x \to a^+$ can be defined directly as follows:

6.2.4. DEFINITION. A distribution f on $I =]a, b[$ **tends to** λ as $x \to a^+$ iff there exists $p \in IN_0$ and $F \in C(I)$, such that:

$$f = D^p F \text{ and } \frac{F(x)}{(x-a)^p} \to \frac{\lambda}{p!} \text{ as } x \to a^+ \text{ (in the ordinary sense).}$$

Besides, the concepts of convergence corresponding to the cases $x \to +\infty$ and $x \to a^+$ are related to each other according to the following rule:

6.2.5. *Suppose* $I =]a, +\infty[$, $\beta > 0$ *and* $f \in \mathscr{D}'(I)$. *Then, if*

$$g(t) = f\left(a + \beta\frac{1}{t}\right), \text{ we have: } \lim_{t \to +\infty} g(t) = \lambda \Leftrightarrow \lim_{x \to a^+} f(x) = \lambda.$$

PROOF. This obviously reduces to the case $a = 0$ and $\lambda = 0$ with $\beta = 1$. Suppose $f(x) \to 0$ as $x \to 0^+$. Then, there exists $p \in IN_0$ and $F \in C(I)$ such that $f = D^p F$ and $\frac{F(x)}{x^p} \to 0$ as $x \to 0^+$. Moreover (cf. 4.5), we have $g(t) = (-t^2 D_t)^p F\left(\frac{1}{t}\right)$ and it is easily shown by induction on p that there exists $p+1$ numbers a_k (whose expression are not needed here) such that

6.2.6.
$$g(t) = \sum_{k=0}^{p} a_k D_t^k \left[t^{p+k} F\left(\frac{1}{t}\right)\right].$$

Now, since $\frac{F(x)}{x^p} \to 0$ as $x \to 0^+$, $t^p F\left(\frac{1}{t}\right) \to 0$ as $t \to +\infty$. Hence

$$\lim_{t \to +\infty} \frac{t^{p+k} F\left(\frac{1}{t}\right)}{t^k} = 0, \quad \text{for } k = 0, \ldots, p,$$

which according to definition 6.1.9. means that all terms on the right side of 6.2.6. $\to 0$ as $t \to +\infty$. In a similar way, we prove that, if $g(t) \to 0$ as $t \to +\infty$, then $f(x) \to 0$ as $x \to 0^+$. ♦

We can obviously *define the concept*:

$$f(x) \to \lambda \text{ as } x \to b^-$$

as we did for the case $x \to a^+$ considering now an interval $]a, b[$, *bounded on the right*. It is readily seen that all preceding propositions and remarks can be extended to this case.

Let I, be now any open interval in $/R$, $I =]a, b[$, and let c be any point of I, that is $a < c < b$. Then if $f \in \mathscr{D}(I)$, we define the concepts:

$$\text{“}f(x) \to \lambda \text{ as } x \to c^+\text{”}$$

$$\text{“}f(x) \to \lambda \text{ as } x \to c^-\text{”}$$

by considering, instead of f, *its restrictions* to the intervals $]a, c[$ and $]c, b[$. As in classical analysis, we shall put

$$f(a^+) = \lim_{x \to a^+} f(x) \text{ (right-hand limit of } f \text{ at } a)$$

$$f(a^-) = \lim_{x \to a^-} f(x) \text{ (left-hand limit of } f \text{ at } a)$$

whenever the limit in question exists.

6.2.7. DEFINITION. We say that f **tends to** λ as $x \to c$ iff $f(x) \to \lambda$ as $x \to c^+$ and $f(x) \to \lambda$ as $x \to c^-$. In this case, we write $\lambda = \lim_{x \to c} f(x)$.

According to preceding definitions and remarks, we can also define directly this concept:

6.2.8. DEFINITION. The distribution f **tends to** λ as $x \to c$ iff there exists an integer $p \geq 0$ and a function F continuous at every point x of I distinct from c, such that:

$$f = D^p F \text{ and } \lim_{x \to c} \frac{F(x)}{(x-c)^p} = \frac{\lambda}{p!} \text{ in the ordinary sense.}$$

6.2.9. Remark. Suppose, more generally, that I is any non-degenerate interval in $/R$ and that c is in the closure of I. Then, definition 6.2.8. applies, *even* if c is a extremity of the domain I of f; for example, if c is a left extremity of I, we have by definition:

$$\lim_{x \to c} f(x) = \lim_{x \to c^+} f(x).$$

With respect to the general hypothesis considered above, we have:

6.2.10. DEFINITION. A distribution f on I is said to be **continuous at a point** c iff there exists $p \in /N_0$ and $F \in C(I)$ such that $f = D^p F$ and $\frac{F(x)}{(x-c)^p}$ is convergent in the ordinary sense as $x \to c$. Then, we write:

$$f(c) = \lim_{x \to c} f(x) = p! \lim_{x \to c} \frac{F(x)}{(x-c)^p}$$

and the number $f(c)$ is said to be the **value of the distribution** f at the point c (or, for $x = c$).

From the linearity property of limits follows:

6.2.11. *If f and g are continuous at c, so is $\alpha f + \beta g$ for $\alpha, \beta \in \mathbb{C}$ and $(\alpha f + \beta g)(c) = \alpha f(c) + \beta g(c)$.*

Examples. 1 – Consider $f(x) = \cos\frac{1}{x}$. Then f is a locally summable function on $/R$ and since:

$$\cos\frac{1}{x} = 2x\sin\frac{1}{x} - D\left(x^2\sin\frac{1}{x}\right),$$

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0,$$

it is easily seen that f is continuous at the point 0 with the value 0 (in distributional sense, and not in ordinary sense!).

2 – It can be seen that $\lim_{x\to 0}\delta^{(k)}=0$, and yet $\delta^{(k)}$ is not continuous at 0 for any $k=0, 1, \ldots$.

3 – It can be proved, as an exercise, that: *If f is a distribution on an interval I minus a point c of $\bar{I}$, and if f is convergent as $x\to c$, then there exists one and only one distribution $\tilde{f}$ on $I\cup\{c\}$, which is continuous at c and such that $\tilde{f}=f$ on I.*

Remark. The previous concepts of limits and value of a distribution at a point of /R have been introduced by Lojasiewicz. As for the concepts of limit as $x\to+\infty$ or as $x\to-\infty$, the definitions given by Mikusinski and Sikorski seem to be to restrictive as they are not invariant for very simple substitutions such as $x=1/t$ and do not allow the justification of certain integral formulas occuring in applications. The definitions that we are using here do not present these inconveniences.

6.3. Primitives and integrals of distributions

If f is a distribution with domain in /R, we call **primitive** of f any distribution φ such that $D\varphi=f$. From this definition follows:

6.3.1. THEOREM. *Every distribution f has infinitely many primitives, and, if the domain of f is an interval then any two primitives of f differ by a constant.*

PROOF. In the general case, the domain of f will be the union of a system of mutually disjoint intervals (cf. 2.5); so we can reduce this to the case of a single interval. Let f be a distribution on I. Then f is of the form $f=D^nF$, with $F\in C(I)$, and every distribution φ of the form $\varphi=D^n\mathfrak{I}F+\mathrm{K}$, where $\mathfrak{I}$ is an integration operator and $K\in\mathbb{C}$, is obviously a primitive of f. Suppose now that $D\varphi_1=D\varphi_2=f$; then if $\varphi_1=D^n\Phi_1$ and $\varphi_2=D^n\Phi_2$, with Φ_1 and Φ_2 in $C(I)$, we have $D^{n+1}\Phi_1=D^{n+1}\Phi_2$, which implies, by axiom 4 (cf. 2.2), that $\Phi_1-\Phi_2$ is a polynomial P of degree $<n+1$. Thus $\varphi_1-\varphi_2=D^nP=$ constant. ♦

From 6.2.2. and 6.2.10. follows immediately:

6.3.2. COROLLARY. *If there exists a primitive of f which is continuous at a point a, then every primitive of f is continuous at a. If, in addition, the domain of f is an interval I, then for every complex number K, there exists one and only one primitive φ of f such that $\varphi(a)=K$.*

It will be natural to denote by the symbol

$$\int_a^{\hat{x}} f(\xi)\,d\xi \quad \text{or shortly by} \quad \int_a^{\hat{x}} f$$

the primitive of f assuming the value 0 at a. (Remember that the sign ^ indicating that x is a dummy variable may be omitted whenever no confusion is possible). Thus according to 6.3.2., *if there exists at least one primitive of f which is continuous at a,* the differential equation $D\varphi=f$ will have a single solution satisfying the initial condition $\varphi(a)=K$, and such a solution is:

$$\varphi(x)=K+\int_a^x f(\xi)\,d\xi.$$

As we have observed, it is understood that here x is only a dummy variable; the distribution φ need not actually have a value $\varphi(x)$ at *every* point x of I. But, obviously, if φ has a value at *some* point b of I, this value is naturally denoted by:

$$\varphi(b)=K+\int_a^b f(\xi)\,d\xi.$$

Thus the integral $\int_a^b f(\xi)\,d\xi$ $\left(\text{in short } \int_a^b f\right)$ is *defined* by the generalized Barrow Formula:

$$\int_a^b f(x)\,dx=\varphi(b)-\varphi(a).$$

Corollary 6.3.2. can be extended as follows:

6.3.3. COROLLARY. *If there exists a primitive of f having a limit as $x\to a^+$ [resp. as $x\to a^-$], then every primitive of f has a limit as $x\to a^+$ [resp. as $x\to a^-$]. If, in addition, the domain of f is an interval I, then for every complex number K, there exists one and only one primitive φ of f such that $\varphi(a^+)=K$ [resp. $\varphi(a^-)=K$].*

Remember that the existence of both $\varphi(a^+)$ and $\varphi(a^-)$ does not imply the existence of $\varphi(a)$.

All preceding remarks and conventions may now be extended to the newly considered cases. For example, we shall denote by

$$\int_{a^-}^x f(\xi)\,d\xi \left(\text{in short } \int_{a^-}^x f\right)$$

the primitive of f on I which tends to zero as $x\to a^-$; accordingly, if such a limit exists, the differential equation $D\varphi=f$ along with the initial condition $\varphi(a^-)=K$ will have the only solution

$$\varphi(x)=K+\int_{a^-}^x f(\xi)\,d\xi.$$

So, we have by definition

$$\int_{a^-}^{b^-} f(x)\,dx=\varphi(b^-)-\varphi(a^-)\,,\quad \int_{a^-}^{b^+} f(x)\,dx=\varphi(b^+)-\varphi(a^-).$$

If $a<b$, these are, respectively, *the integral of the distribution* f *on the intervals* $[a, b[$ *and* $[a, b]$. The integrals of f on $]a, b]$ and $]a, b[$ are analogously defined. Naturally such an integral is said to *exist* or to be *convergent* iff the two corresponding limits exist. If $b \leq a$, we have of course:

$$\int_{a^-}^{b^+} f = -\int_{b^+}^{a^-} f\,, \quad \int_{a^+}^{b^+} f = -\int_{b^+}^{a^+} f\,, \text{ etc.}$$

Finally, all preceding definitions may be extended to *infinite intervals.* For example, we have by definition

$$\int_{a^-}^{+\infty} f(x)\,dx = \varphi(+\infty) - \varphi(a^-),$$

if φ is a primitive of f such that the limits on the right-hand side exist; and $\int_{a^-}^{+\infty} f(x)\,dx$ is called the integral of f on the interval $[a, +\infty[$. For other kinds of infinite intervals the definitions are quite analogous.

In the general case, a distribution f is said to be **integrable over an interval** I, iff the integral of f on I exists. This integral may be denoted by $\int_I f(x)\,dx$ or simply by $\int_I f$.

From the linearity property of limits follows immediately the corresponding property for integrals:

6.3.4. LINEARITY PROPERTY. *If two distributions f and g are integrable over I, so is $\alpha f + \beta g$ for any $\alpha, \beta \in \mathbb{C}$ and*

$$\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g\,.$$

On the other hand it should be observed that:

6.3.5. *If f is a function summable on I, then the integral of f over I, in the distributional sense, exists and equals the Lebesgue integral over I. More generally, if f is a locally summable function on I such that* $\int_I f$ *is convergent in the classical sense (even simply convergent), then* $\int_I f$ *exists, in the distributional sense, and has the same value.*

However, the converse of this proposition is not true, as we shall presently see:

Examples. 1 – Consider the integral $\int_I f(x)\delta^{(n)}(x-a)$ where I is any interval in $/R$, n an integer ≥ 0 and $f \in C^n$ a function on I. Then:

$$f\delta^{(n)}(\hat{x}-a)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}D^{n-k}[f^{(k)}(a)\delta(\hat{x}-a)].$$

Now, for every $k<n$, a primitive of $D^{n-k}[f^{(k)}(a)\delta(\hat{x}-a)]$ is the distribution $f^{(k)}(a)\delta^{(n-k-1)}(\hat{x}-a)$ which tends to zero as x tends to any point x_0 in $/R$. Hence:

$$\int_I f(x)\delta^{(n)}(x-a)\,dx=(-1)^n f^{(n)}(a)\int_I \delta(x-a)\,dx=\begin{cases}(-1)^n f^{(n)}(a)\ , & \text{if } a\in I\\ 0 & , \text{if } a\notin I.\end{cases}$$

For example:

$$\int_{a^-}^{a^+} f(x)\delta''(x-a)\,dx = f''(a)\int_{a^-}^{a^+}\delta(x-a)\,dx = f''(a).$$

2 – Consider the integral $\int_{/R} e^{i\omega t}dt$, where ω is a real parameter. This integral is obviously divergent, in the classical sense, for every value of ω. However, for $\omega \neq 0$, one primitive of $e^{i\omega t}$ is $\dfrac{e^{i\omega t}}{i\omega}$ and

$$\frac{e^{i\omega t}}{i\omega} = \frac{1}{(i\omega)^2} De^{i\omega t} , \quad \lim_{t\to\infty} \frac{e^{i\omega t}}{t} = 0.$$

Hence, we have, in the distributional sense, for every $\omega \neq 0$:

$$\int_{-\infty}^{+\infty} e^{i\omega t} dt = \frac{1}{i\omega} \left(\lim_{t\to+\infty} e^{i\omega t} - \lim_{t\to-\infty} e^{i\omega t} \right) = 0.$$

For $\omega=0$, this integral is divergent, even in the distributional sense. This result agree with the intuition of physicists, which have, long since, adopted the formula:

$$\int_{/R} e^{i\omega t} dt = 2\pi\delta(\omega).$$

However, a complete justification of this formula cannot be achieved, without a suitable definition of parametric integral, which will be given in chapter VIII.

The case considered in example 1 is included in the following proposition:

6.3.6. *Every distribution with a bounded carrier on /R is integrable on /R*.

PROOF. Let f be a distribution of bounded carrier on $/R$. This means that there exists a bounded interval $I=[a, b]$ such that $f=0$ outside I. Hence, if φ is a primitive of f, $D\varphi=0$ outside I and φ reduces to constants c_1 and c_2, respectively, on $]-\infty, a[$ and on $]b, +\infty[$. Thus $\varphi(-\infty)=\varphi(a^-)=c_1$ and $\varphi(b^+)=\varphi(+\infty)=c_2$. Hence, f is integrable on $/R$ and

$$\int_{/R} f = \int_I f = \int_{a^-}^{b^+} f = c_2 - c_1. \ \blacklozenge$$

A complementary proposition to 6.3.6., which can be proved in a similar way is the following:

6.3.7. *Whenever f is integrable on $\mathbb{R}$, we have $\int_{\mathbb{R}} f = \int_I f$, for every interval containing the carrier of f.*

For example, if f is integrable on $\mathbb{R}$ and zero for $x<a$, then

$$\int_{\mathbb{R}} f = \int_{a^-}^{+\infty} f.$$

In order to obtain more powerful tests for the convergence of integrals, we are going to develop the concept of order of growth for distributions.

6.4. Orders of growth for distributions

For brevity, we shall confine ourselves to the typical case where $x \to +\infty$, since the considerations in the other cases are analogous. Let I be any interval *unbounded on the right* and $\Im f(x) = \int_c^x f$, with $c \in I$, for $f \in C(I)$. The extension of the symbol "O" to distributions is based on the following lemma, whose proof is similar to the one of 6.1.3. and even more simple:

6.4.1. LEMMA. *If f is a continuous function on I such that $f \in O(x^\alpha)$ as $x \to +\infty$, with $\alpha > -1$, then $\Im f \in O(x^{\alpha+1})$ as $x \to +\infty$.*

6.4.2. DEFINITION. If $f \in \mathscr{D}(I)$ and $\alpha > -1$, then we write $f \in O(x^\alpha)$ as $x \to +\infty$ iff there exist $n \in \mathbb{N}_0$ and $F \in C(I)$, such that $f = D^p F$ and $\dfrac{F(x)}{x^{n+\alpha}}$ is bounded on the right.

The lemma guarantees the linearity property for this case. In particular:

6.4.3. DEFINITION. A distribution f on I is said to be **bounded on the right** iff $f \in O(1)$ as $x \to +\infty$, that is iff there exist $n \in I\!N_0$ and $F \in C(I)$ such that $f = D^n F$ and $\frac{F(x)}{x^n}$ is bounded on the right.

That being so, we are able to define the meaning of the expression "$f \in o(\varphi)$" and "$f \in O(\varphi)$" in the more general case when $f \in \mathscr{D}(I)$ and $\varphi \in C^\infty(I)$. For all that purpose, we can take as a model the classical definition 6.1.1.:

6.4.4. DEFINITION. We shall write $f \in o(\varphi)$ as $x \to +\infty$ iff there exists a real x_0 and a distribution f_0 such that:

$$f = \varphi f_0 \quad \text{for} \quad x > x_0 \quad \text{and} \quad f_0 \to 0 \quad \text{as} \quad x \to +\infty.$$

We shall write $f \in O(\varphi)$ as $x \to +\infty$ iff there exists a real x_0 and a distribution f_0 such that $f = \varphi f_0$ for $x > x_0$ and f_0 is bounded on the right.

The first thing to do is to see whether these definitions are equivalent to the preceding ones in the particular case, when φ is of the form x^α, with $\alpha > -1$. This equivalence is easily proved by means of the formulas:

$$x^\alpha D^n F_0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} D^{n-k}(F_0 D_x^k x^\alpha)$$

$$D^n(x^\alpha G_0) = \sum_{k=0}^{n} \binom{n}{k} (D_x^k x^\alpha) D^{n-k} G_0$$

taking into account the linear property.

On the other hand, this same property can be now immediately extended to the general case. Moreover definition 6.4.4. introduce a remarkable new property which is a counterpart of the preceding lemmas.

6.4.5. DIFFERENTIATION PROPERTY. *If $f \in O(x^{\alpha})$ on the right, then $Df \in O(x^{\alpha-1})$ on the right, for every $\alpha \in I\!R$.*

We shall begin the proof in the case $\alpha = 0$:

6.4.6. *If f is bounded on the right, then $Df \in O(x^{-1})$ as $x \to +\infty$.*

Suppose f bounded on the right. Then, there exists $p \in I\!N_0$, $F \in C(I)$ and c such that $f = D^p F$ for $x > c$ and $\dfrac{F(x)}{x^p}$ is bounded on the right. We may choose $c > 0$; then we have:

$$Df = x^{-1}(xD^{p+1}F) = x^{-1}[D^{p+1}(xF) - (p+1)D^p F] \text{ for } x > c$$

and it is readily seen that $D^{p+1}(xF)$ is bounded on the right, as well as $D^p F$. Hence $Df \in O(x^{-1})$ as $x \to +\infty$.

Suppose now $f \in O(x^{\alpha})$ $x \to +\infty$, where $\alpha \in I\!R$. Then there exist x_0 and f_0 such that $f = x^{\alpha} f_0$ for $x > x_0$ and $f_0 \in O(1)$ on the right. It follows that $Df = \alpha x^{\alpha-1} f_0 + x^{\alpha} Df_0$ and it is readily seen, applying 6.4.6., that $Df \in O(x^{\alpha-1})$ as $x \to +\infty$. ♦

By an identical argument, it is shown that the *differantiation property extends to the "o" symbol.*

Furthermore it is a simple matter to prove the following properties where the expression "on the right" or "as $x \to +\infty$" is omitted for simplicity.

6.4.7. *If f is convergent, then f is bounded.*

6.4.8. *If $f \in o(\varphi)$ then $f \in O(\varphi)$.*

6.4.9. *If $f \in O(x^{\alpha})$ and $\alpha < \beta$, then $f \in o(x^{\beta})$.*

Obviously we have chosen the case when $x \to +\infty$ as a model; the concepts and properties are quite analogous in cases such as $x \to -\infty$, $x \to c^+$, etc..

6.4.10. Convention. If a distribution f has the same growth property as $x \to +\infty$ and as $x \to -\infty$, we shall say that f has this property as $x \to \infty$. If $f \in \mathscr{D}(I)$ is bounded on the right and on the left (respectively as x tends to the right extremity and to the left extremity of I), we shall say that f is **bounded on** I or simply **bounded**.

Remark. The concept of bounded distribution that we have just introduced is more general than the concept of bounded distribution according to Schwartz and necessary for the integral theory as we shall next see.

6.5. Convergence tests for integrals

Let us consider, at first, the case of integrals on $I\!R$. We have the following test, which is not true in classical analysis:

6.5.1. (A NECESSARY CONDITION FOR CONVERGENCE). *If a distribution f is integrable on $I\!R$, then $f \in O(x^{-1})$ as $x \to \infty$.*

PROOF. Suppose there exists a primitive φ of f such that φ is convergent as $x \to +\infty$ and as $x \to -\infty$[6]. Then by 6.4.7., φ is bounded on $I\!R$ and, by 6.4.6. (and its analog for the case $x \to -\infty$) we have $D\varphi \in O(x^{-1})$ *as* $x \to \infty$. ♦

The following theorem extends to distributions a well known classical test.

6.5.2. (A SUFFICIENT CONDITION FOR CONVERGENCE). *If there exists a number $\alpha < -1$ such that $f \in O(x^{\alpha})$ as $x \to \infty$, then f is integrable on $I\!R$.*

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(6) – This does not mean that φ is convergent as $x \to \infty$, for the limits are in general different.

PROOF. Suppose $f\in O(x^{\alpha})$ as $x\to\infty$ with $\alpha<-1$. Then there exists a number $c>0$, an integer $n\geq 0$ and a continuous function F such that:

$$f=x^{\alpha}D^{n}F \text{ for } |x|>c, \text{ with } \frac{F(x)}{x^{n}} \text{ bounded for } |x|>c.$$

Set: $$F_1(x)=\begin{cases}F(x), & \text{for } |x|>c\\ 0, & \text{for } |x|<c\end{cases},\quad f_1=x^{\alpha}D^{n}F_1,\quad f_2=f-f_1.$$

Then f_2 is a distribution with carrier contained in $[-c,+c]$; hence integrable on $/R$ (cf. 6.3.6.). So we have only to prove that f_1 is integrable on $/R$, for then we have:

$$\int_{/R} f=\int_{/R} f_1+\int_{/R} f_2.$$

We shall put $f_1=f$ and $F_1=F$. Then:

$$f=x^{\alpha}D^{n}F=\sum_{k=0}^{n}(-1)^{k}c_{k}D^{n-k}(x^{\alpha-k}F)$$

where $c_k=\alpha(\alpha-1)\cdots(\alpha-k+1)\binom{n}{k}$. From here we deduce the following primitive of f:

6.5.3. $$\varphi=\sum_{k=0}^{n-1}(-1)^{k}c_{k}D^{n-k-1}(x^{\alpha-k}F)+(-1)^{n}c_{n}\int_{0}^{x}\xi^{\alpha-n}F(\xi)\,d\xi.$$

But since $F\in O(x^{n})$ as $x\to\infty$ in the ordinary sense, we have $\xi^{\alpha-n}F\in O(\xi^{\alpha})$ as $x\to\infty$ with $\alpha<-1$ and, according to the classical test, this implies that the primitive $\xi^{\alpha-n}F$ is summable on $/R$. Hence the last term in 6.5.3. is convergent as $x\to+\infty$ and as $x\to-\infty$.

As to the other terms, observe that the functions

$$\frac{x^{\alpha-k}F(x)}{x^{n-k-1}}=x^{\alpha+1}\frac{F(x)}{x^n} \text{ for } k=0,\ldots,n-1,$$

tend to zero as $x\to\infty$ since $\alpha+1<0$ and $\dfrac{F(x)}{x^n}$ is bounded (in the ordinary sense). Hence, by definition 6.1.9.

$$D^{n-k-1}(x^{\alpha-k}F)\to 0 \text{ as } x\to\infty,$$

so that $\varphi(+\infty)=(-1)^n c_n\int_0^{+\infty} x^{\alpha-n}F,\qquad \varphi(-\infty)=(-1)^n c_n\int_0^{-\infty} x^{\alpha-n}F.$

Therefore f is integrable on $/R$ and

$$\int_{/R} f=\int_{/R} x^\alpha D^n F=(-1)^n c_n\int_{/R} x^{\alpha-n}F=(-1)^n\int_{/R} FD^n x^\alpha. \ \blacklozenge$$

We can deduce similar tests for integrals on intervals distinct from $/R$. For example, consider an interval $I=]a,+\infty[$ and $f\in\mathscr{D}(I)$. Then it is easily seen that *if f is integrable in I, then $f\in O(x^{-1})$ as $x\to+\infty$ and $f\in O((x-a)^{-1})$ as $x\to a^+$. If there exists $\alpha<-1$ and $\beta>-1$ such that $f\in O(x^\alpha)$ as $x\to+\infty$ and $f\in O((x-a)^\beta)$ as $x\to a^+$, then f is integrable on I.*

6.6. Multiplication and change of variables in connection with limits and integrals

It is a simple matter to prove the following propositions:

6.6.1. *If $f\in\mathscr{D}(I)$ is convergent as $x\to c^+$ with $c\in I$ and if $g\in C^\infty(I)$ then fg is convergent as $x\to c^+$ and:*

$$\lim_{x\to c^+}(fg)=\left(\lim_{x\to c^+} f\right)\left(\lim_{x\to c^+} g\right).$$

6.6.2. *If $f \in \mathscr{D}(I)$ is convergent as $x \to c^+$ with $c \in I$ and if h is a C^∞ mapping of an interval I^* into I, such that $h'(t) > 0$ in I^*, then $f(h(t))$ is convergent as $t \to \gamma^+$ with $h(\gamma) = c$ and:*

$$\lim_{t \to \gamma^+} f(h(t)) = \lim_{t \to c^+} f(x).$$

Obviously, these two propositions can be extended to the case when f is convergent as $x \to c^-$. Then the second one enable the usual substitution property to be extended to the integrals of distributions on bounded intervals. For example, assuming $f \in \mathscr{D}(I)$, $a, b \in I$ and h is an increasing C^∞ mapping of I^* into I such that $a = h(\alpha)$, $b = h(\beta)$, we have

$$\int_{a^+}^{b^-} f(x)\,dx = \int_{\alpha^+}^{\beta^-} f(h(t))h'(t)\,dt$$

whenever the first integral exist.

However these criterions are not sufficient in certain cases which occur in practice. Our next purpose is to introduce a stronger criterium than 6.6.2.. For simplicity, we shall reduce our discussion to the case where $x \to +\infty$ and $h(+\infty) = +\infty$, which can be taken as a model for other cases.

6.6.3. THEOREM. *Let $f \in \mathscr{D}(I)$, I unbounded on the right, and let h be a C^∞ mapping of an interval I^* into I such that $h'(t) \neq 0$ on I^* and $h(t) \to +\infty$ as $t \to +\infty$. Suppose that:*

(i) f is convergent as $x \to +\infty$

(ii) *h' tends to a number $c \neq 0$ as $t \to +\infty$ (in the ordinary sense)*

(iii) *$h^{(k)} \in o(t^{-k+1})$ as $t \to +\infty$ (in the ordinary sense), for $k > 1$.*

Then we have: $\lim\limits_{t \to +\infty} f(h(t)) = \lim\limits_{x \to +\infty} f(x)$

PROOF. Suppose $f \to \lambda$ as $x \to +\infty$. Then there exist $n \in \mathbb{N}_0$ and $F \in C(I)$ such that $f = D^n F$ and $\dfrac{F(x)}{x^n} \to \dfrac{\lambda}{n!}$ as $x \to +\infty$. Now

$f\circ h=\left(\frac{1}{h'}D_t\right)^n(F\circ h)$ and according to the hypothesis:

$$\lim_{t\to+\infty}\frac{h(t)}{t}=\lim_{t\to+\infty}h'(t)=c.$$

Hence:

6.6.3'. $$\frac{F(h(t))}{t^n}=\frac{F(h(t))}{(h(t))^n}\left(\frac{h(t)}{t}\right)^n\to\frac{\lambda c^n}{n!}.$$

On the other hand it is easily seen that:

$$\left(\frac{1}{h'}D_t\right)^n(F\circ h)=\sum_{k=0}^{n}D_t^{n-k}\left(\alpha_k(t)F(h(t))\right)$$

where $\alpha_0=\left(\frac{1}{h'}\right)^n$ and $\alpha_k\in o(t^{-k})$ *as* $t\to+\infty$, for $k=1, 2,\dots, n$. Thus all terms in the last sum tend to zero as $x\to+\infty$, except $D_t^n(\alpha_0(F\circ h))$, which, by 6.6.3', tends to λ. ♦

This criterium and the corresponding ones for the cases when $x\to-\infty$, $t\to-\infty$, etc., lead to the following substitution rule for integrals.

6.6.4. COROLLARY. *Let f be a distribution integrable on IR and h a C^∞ mapping of IR onto IR such that:*

(j) $h'(t)$ is $\neq 0$ on IR and tends to numbers $\neq 0$ as $t\to+\infty$ and as $t\to-\infty$ (in the ordinary sense)

(jj) $h^{(k)}\in o(t^{-k+1})$ as $t\to\infty$ (in the ordinary sense) for all $k=2, 3,\dots$.

Then $f(h(t))$ is integrable on IR and:

$$\int_{IR}f(x)\,dx=\int_{IR}f(h(t))|h'(t)|\,dt.$$

This rule is an immediate consequence of theorem 6.6.3. and its corresponding theorems applied to a primitive φ of f. Observe that, in the case $h'(t)<0$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{+\infty}^{-\infty} f(h(t))h'(t)\,dt = -\int_{-\infty}^{+\infty} f(h(t))h'(t)\,dt\,.$$

In particular 6.6.4. applies in the elementary cases when $x=t+a$ or $x=ct$, with $a\in IR$ and $c\in\mathbb{C}$. Then we have:

6.6.5. $$\int_{IR} f(x)\,dx = |c|\int_{IR} f(cx)\,dx\,.$$

6.6.6. $$\int_{IR} f(x+a)\,dx = \int_{IR} f(x)\,dx\,.$$

The last formula can be expressed by saying that the integral is invariant under translations.

More refined criterions can be obtained by using the concept of measure as we did for multiplication in chapter IV.

Remember that if μ is a measure on an open interval I, the **total variation** of μ in a bounded interval J such that $\bar{J}\subset I$ is defined to be the supremum of the sums $S_p=\sum_{1}^{p}|\mu(J_i)|$, for all finite partitions P of J into intervals J_1, ..., J_p. We shall denote by $|\mu|(J)$ the total variation of μ in J; as is well known, $|\mu|$ is again a measure on IR (the **modulus** of μ) such that:

(i) if $\mu\in\mathring{L}$, then $|\mu|$ is the modulus of the function μ in the ordinary sense;

(ii) $|\varphi\mu|=|\varphi||\mu|$ for all $\varphi\in C(I)$.

On the other hand, if μ and ν are two measures on I, we write $\mu\leq\nu$ iff $\mu(J)\leq\nu(J)$ for all bounded intervals J such that $\bar{J}\subset I$.

Suppose I is unbounded on the right. A measure μ on I is said to be bounded on the right if and only if there exist two numbers x_0 and k such that $|\mu|\leq k$ for $x>x_0$; i.e. $|\mu|(J)\leq k|J|$ for all bounded intervals $J\subset[x_0, +\infty[$. On the other hand, we say that μ **converges to**

a number c as $x \to +\infty$, iff for every $\varepsilon > 0$, there exists a real x_0 such that $|\mu - c| < \varepsilon$ for $x > x_0$. It is readily seen that these concepts coincide with the classical ones if μ is a function. Besides, the preceding lemma for the "o" and "O" symbols keep true if f is a measure.

These remarks suggest the following refinement of the concept of convergence for distributions:

6.6.7. DEFINITION. Let $f \in \mathscr{D}'(I)$, $n \in I\!N_0$ and $\lambda \in \mathbb{C}$. We write $f \underset{n}{\longrightarrow} \lambda$ as $x \to +\infty$ if and only if there exist a real x_0 and a measure F such that $f = D^n F$ and $\dfrac{F(x)}{x^n} \to \dfrac{\lambda}{n!}$ (in measure sense) as $x \to +\infty$. On the other hand, if $\varphi \in C^\infty(I)$, we shall write $f \in o_n(\varphi)$ as $x \to +\infty$ iff there exists x_0 and f_0 such that $f = \varphi f_0$ for $x > x_0$ and $f_0 \underset{n}{\longrightarrow} 0$ as $x \to +\infty$.

The expression "$f \in O_n(\varphi)$" can be analogously defined and the "dual" concepts of the preceding ones can be introduced as follows:

6.6.8. DEFINITION. Let $n \in I\!N_0$, $f \in C^n(I)$ and $\lambda \in \mathbb{C}$. We shall write $f \overset{n}{\longrightarrow} \lambda$ as $x \to +\infty$ iff f tends to λ and $f^{(k)} \in o(x^{-k})$ as $x \to +\infty$, for $k = 1, \ldots, n$ (in the ordinary sense). We write $f \in o^n(\varphi)$ as $x \to +\infty$ iff there exists x_0 and f_0 such that $f = \varphi f_0$ for $x > x_0$ and $f_0 \overset{n}{\longrightarrow} 0$ as $x \to +\infty$.

Thus, it is readily seen that:

6.6.9. *If $f \underset{n}{\longrightarrow} \lambda$ as $x \to +\infty$ and $g \overset{n}{\longrightarrow} \mu$ as $x \to +\infty$, then $fg \underset{n}{\longrightarrow} \lambda\mu$ as $x \to +\infty$.*

6.6.10. *If $f \underset{n}{\longrightarrow} \lambda$ as $x \to +\infty$ and if h is a C^n mapping of an interval I^* into I such that $h \to +\infty$ as $t \to +\infty$ and $h' \overset{n}{\longrightarrow} c$, with $c \neq 0$ and $c \neq \infty$, then $f \circ h \underset{n}{\longrightarrow} \lambda$ as $t \to +\infty$.*

6.6.11. *If $f \in o_n(\varphi)$ and $g \in o^n(\psi)$ on the right, then $fg \in o_n(\varphi\psi)$ on the right, and analogously for the "O" symbol.*

6.7. Scalar products. Definition of distributions according to Sobolev-Schwartz

We shall say the two distributions f and g are multipliable if and only if the product fg exists in some of the senses considered in chapter IV. That being so:

6.7.1. DEFINITION. If two distributions f and g on an interval I are multipliable and fg is integrable on I, then $\int_I fg$ will be called the **symmetrical scalar product** or simply the **scalar product** of f by g and is denoted by $\langle f, g\rangle$:

$$\langle f, g\rangle = \int_I fg.$$

Obviously, the scalar product is in fact *symmetrical* (or *commutative*). Moreover it is *bilinear*: for all $\lambda, \mu \in \mathbb{C}$, we have

$$\langle \lambda f_1 + \mu f_2, g\rangle = \lambda\langle f_1, g\rangle + \mu\langle f_2, g\rangle$$

whenever $\langle f_1, g\rangle$ and $\langle f_2, g\rangle$ exist and analogously on the right.

Observe that every distribution f on I can be written in the form $f = u + iv$, where u, v are *real-valued distributions* (i.e., of the form $u = D^n U$, $v = D^n V$, where U and V are real-valued continuous functions on I). Then we put $\bar{f} = u - iv$ (*conjugate of* f). It is readily seen that if $\langle f, g\rangle$ exists, then $\langle f, \bar{g}\rangle$ exists also.

We call $\int_I f\bar{g}$ the **hermitic scalar product** or simply the **hermitic product** of f by g, and it is denoted by $\langle f | g\rangle$:

$$\langle f | g\rangle = \int_I f\bar{g}.$$

The hermitic product is not commutative:

$$\langle f|g\rangle=\overline{\langle g|f\rangle};$$

but it is of course, linear on the left.

Remember that any function $f\in L^2(I)$ (square summable function on I) is locally summable, hence a distribution. It is well known that if $f, g\in L^2(I)$, then $fg\in L(I)$ so that $\langle f|g\rangle$ is a hermitic form on $L^2(I)$, which makes $L^2(I)$ a Hilbert space. The following is a classical theorem in functional analysis.

6.7.2. *If E is a Hilbert space, there is a one-to-one correspondence between the continuous linear functionals on E and the elements of E. The functional U corresponding to an element u of E is given by the formula:*

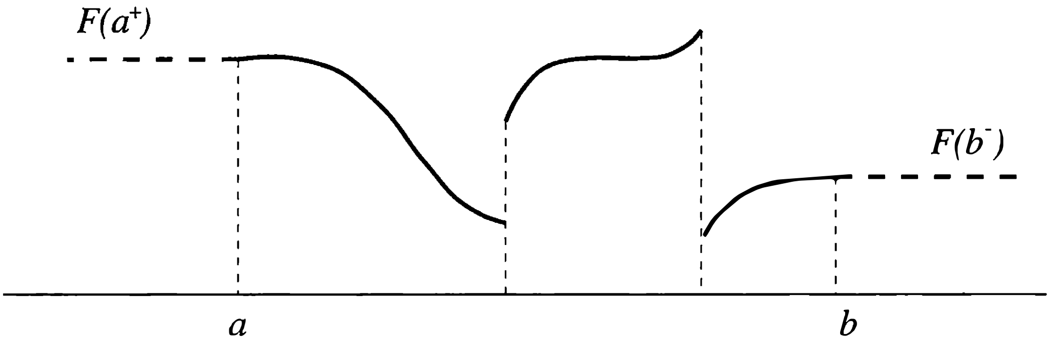
$$U(x)=\langle x|u\rangle \textit{ for all } x\in E.$$

Moreover, this correspondence is a vector isomorphism between E and E'. But the elements of E' (**covariant** vectors) do not behave like the elements of E (**contravariant** vectors) by change of bases; thus *it is not convenient in most cases to identify E' with E.*

For developing the study of scalar distributions, a remark about terminology is necessary. When I is a compact interval, the expression "measure on I" is commonly used with a meaning equivalent to that of "measure of an interval contained in I". For example, in this sense, δ may be considered as a measure on $I=[0, 1]$; *but the restriction of the δ distribution to* $[0, 1]$ *is* $D(\rho_I H)=0$. To avoid confusion, we shall say "measure in I" instead of "measure on I" for a distribution f of the form $f=DF$, when F is a standardized function of bounded variation on I. On the other hand, we shall denote by $M^*(I)$ the vector space of all measures on $/R$ which vanish outside I and by $M(I)$ the set of all measures on I. Observe that:

6.7.3. *If $I=[a, b]$, every measure μ in I can be uniquely extended as a measure $\widetilde{\mu}\in M^*(I)$ such that $\widetilde{\mu}[a, a]=\widetilde{\mu}[b, b]=0$.*

In fact, if $\mu \in M(I)$, then $\mu = DF$ where F is a function of bounded variation on I. Now F can be uniquely extended to a function $\widetilde{F}$ of bounded variation on $/R$ such that $\widetilde{F}(x) = F(a^+)$ for $x < a$ and $\widetilde{F}(x) = F(b^-)$ for $x > b$.



Hence, if we put $\widetilde{\mu} = D\widetilde{F}$, we have $\widetilde{\mu} \in M^*(I)$, $\widetilde{\mu} = [a, a] = = \widetilde{\mu}[b, b] = 0$, and it is readily seen that $\widetilde{\mu}$ is uniquely determined by μ. We call $\widetilde{\mu}$ the **minimal extension** of μ to $/R$.

Another classical theorem in functional analysis is the following:

6.7.4. F. RIESZ THEOREM. *There is a one-to-one correspondence between the measures $f \in M^*(I)$ and the continuous linear functionals u on $C(I)$. This correspondence $f \leftrightarrow u$ is given by:*

$$u(\varphi) = \langle f, \varphi \rangle = \int_I f\varphi, \quad \forall \varphi \in C(I).$$

We are going to deduce some important consequences from this formula. We shall denote by $M_n^*(I)$ the set of all distributions of order $\leq n$ on $/R$ vanishing outside I. Suppose $I = [a, b]$; then

6.7.5. *Every distribution $f \in M_n^*(I)$ can be written in the form:*

$$f = D^n F_0 + \sum_{k=0}^{n-1} c_k \delta^{(k)}(x-a)$$

where $F_0 \in M^*(I)$ and $c_0, \ldots, c_{n-1} \in \mathbb{C}$.

PROOF. Consider $f \in M_n^*(I)$. Then f is of the form $f = D^n F$ where $F \in M(/R)$. On the other hand, since $f = 0$ outside I, F reduces to polynomials P and P^* of degree $<n$, respectively on the left and on the right of I. Put $\widetilde{F}=F-P^*$; then $\widetilde{F}=0$ for $x>b$ and $f=D^n\widetilde{F}$. We can suppose that $\widetilde{F}=F=0$ for $x>b$. Set:

$$P_0=\begin{cases} P & \text{for } x<a \\ 0 & \text{for } x\geq a \end{cases} \qquad F_0=\begin{cases} 0 & \text{for } x<a \\ F & \text{for } x\geq a. \end{cases}$$

Then $F=F_0+P_0$ and $F_0\in M^*(I)$. On the other hand, P_0 is a polynomial of degree $<n$ for $x<0$ and zero for $x>0$, so that D^nP_0 is of the form:

$$\sum_{k=0}^{n-1} c_k\delta^{(k)}(x-a).$$

Consequently

$$f=D^nF_0+\sum_{k=0}^{n-1} c_k\delta^{(k)}(x-a). \ \blacklozenge$$

Let now φ be any C^n function on $/R$ and $f\in M_n^*(I)$. Then, of course, $\varphi f\in M_n^*(I)$; so that φf is integrable on $/R$. Put

$$f=D^n\mu+\sum_{k=0}^{n-1} c_k\delta^{(k)}(x-a) \text{ with } \mu\in M^*(I).$$

A primitive of $\varphi D^n\mu$ will then be:

$$\Phi=\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}D^{n-k-1}(\varphi^{(k)}\mu)+(-1)^n\int_{a^-}^{x^+}\varphi^{(n)}\mu$$

and since $\mu=0$ outside I, we have:

$$\int_I \varphi D^n\mu=\Phi(b^+)-\Phi(a^-)=(-1)^n\int_{a^-}^{b^+}\varphi^{(n)}\mu.$$

On the other hand (cf. 6.3.6. and example 1 in 6.3.)

$$\langle \varphi, \delta^{(k)}(x-a)\rangle=(-1)^k\varphi^{(k)}(a).$$

Hence:

6.7.6. $$\langle f, \varphi\rangle = \sum_{k=0}^{n} (-1)^k c_k \varphi^{(k)}(a) + (-1)^n \int_I \varphi^{(n)} \mu .$$

Observe that in this formula the values of $\varphi(x)$ for $x \notin I$ do not matter. So we may extend this formula by definition to all functions $\varphi \in C^n(I)$. In $C^n(I)$ it is common to define the norm by:

6.7.7. $$\|\varphi\|^n = \sup_{x \in I} \{|\varphi(x)|, |\varphi'(x)|, \ldots, |\varphi^{(n)}(x)|\}.$$

Then $C^n(I)$ becomes a Banach space and the convergence of a sequence (φ_p) to 0 in this norm means the convergence of the n sequences (φ_p), $(\varphi_p^{(1)}), \ldots, (\varphi_p^{(n)})$ to 0 uniformly on I. Now we have the following consequence of the Riesz theorem:

6.7.8. THEOREM. *There is an isomorphism $f \leftrightarrow u$ between the vector spaces $M_n^*(I)$ and $C^n(I)'$ defined by $u(\varphi) = \langle f, \varphi\rangle$ $\forall \varphi \in C^n(I)$.*

PROOF. a) Take $f \in M_n^*(I)$. Then f is of the form:

$$f = D^n \mu + \sum_{k=0}^{n-1} c_k \delta^{(k)}(x-a)$$

with $\mu \in M^*(I)$ and:

$$u(\varphi) = \sum_{k=0}^{n} (-1)^k \varphi^{(k)}(a) + (-1)^n \int_I \varphi^{(n)} \mu .$$

This defines clearly a linear functional u on $C^n(I)$. On the other hand, we have:

$$|u(\varphi)| \leq \sum_{0}^{n-1} |c_k| \, \|\varphi\|^n + \|\varphi\|^n |\mu|(I)$$

which shows that $u(\varphi) \to 0$ as $\varphi \to 0$.

b) Take $u \in C^n(I)'$ and set, for all $\psi \in C(I)$, $v(\psi) = (-1)^n u(\Im^n \psi)$, with $\Im \psi(x) = \int_a^x \psi(\xi)\, d\xi$. Then v is a continuous linear functional on

$C(I)$ and by 6.7.3., there exists $\mu \in M^*(I)$ such that $v(\psi)=\langle \mu, \psi \rangle$. Besides, every function $\varphi \in C^n(I)$ can be written in the form:

$$\varphi(x)=\sum_{k=0}^{n-1} \frac{\varphi^{(k)}(a)}{k!}(x-a)^k+\mathfrak{I}^n\varphi^{(n)}, \quad \text{with } \varphi^{(n)}\in C(I).$$

Hence, if we set $c_k=(-1)^k u\left(\frac{(\hat{x}-a)^k}{k!}\right)$ we find:

$$u(\varphi)=\sum_{k=0}^{n}(-1)^k c_k \varphi^{(k)}(a)+(-1)^n \int_I \varphi^{(n)}\mu$$

since $u(\mathfrak{I}^n\varphi^{(n)})=(-1)^n v(\varphi^{(n)})=(-1)^n\langle u, \varphi^{(n)}\rangle$.

So if we put $f=\sum_0^{n-1} c_k \delta^{(k)}(x-a)+D^n\mu$, we have $u(\varphi)=\langle f, \varphi \rangle$ for all $\varphi \in C^n(I)$. ♦

We shall denote by $C_*^n(I)$ the set of all C^n functions φ on I such that $\varphi^{(k)}(a)=\varphi^{(k)}(b)=0$ for $k=0,\dots, n$. It is obvious that $C_*^n(I)$ is a vector subspace of $C^n(I)$. Also, every $\varphi \in C_*^n(I)$ can be uniquely extended as a C^n function on $/R$ vanishing outside I, so that $C_*^n(I)$ can also be identified with a subspace of $C^n(/R)$. We shall consider $C_*^n(I)$ provided with the norm $\|\cdot\|^n$ defined by 6.7.7. On the other hand $M_n(I)$ is the vector space of all distributions f on I of the form $f=D^n\mu$ with $\mu \in M(I)$. Now from 6.7.8., follows:

6.7.9. THEOREM. *There is an isomorphism $f \leftrightarrow g$ between $M_n(I)$ and $C_*^n(I)'$ defined by $u(\varphi)=\langle f, \varphi \rangle$ for all $\varphi \in C_*^n(I)$. Besides $\langle f, \varphi \rangle=(-1)^n\langle \mu, \varphi^{(n)}\rangle$ if $f=D^n\mu$.*

PROOF. a) Take $f=D^n\mu$ where $\mu \in M(I)$. Then if we set $\widetilde{f}=D^n\widetilde{\mu}$ where $\widetilde{\mu}$ is the minimal extension of μ to $/R$ (cf. 6.7.3.), f defines a functional $\widetilde{\mu}\in C^n(I)'$ whose restriction to $C_*^n(I)$ is obviously an element u of $C_*^n(I)'$ such that

$$u(\varphi)=(-1)^n\int_{a^-}^{b^+}\varphi^{(n)}\widetilde{\mu} \quad \text{for all} \quad \varphi \in C_*^n(I).$$

But as $\widetilde{\mu}[a, a] = \widetilde{\mu}[b, b] = 0$, it is easily seen that

$$\int_{a^-}^{b^+} \varphi^{(n)} \widetilde{\mu} = \int_{a^+}^{b^-} \varphi^{(n)} \widetilde{\mu} .$$

So, we can write $u(\varphi) = \langle f, \varphi \rangle = (-1)^n \langle u, \varphi^{(n)} \rangle$.

b) Take now $u \in C_*^n(I)'$. Observe that for every function $\varphi \in C^n(I)$ there is one and only one function $\varphi_0 \in C_*^n(I)$ such that:

$$\varphi_0(x) = \varphi(x) - \sum_{k=0}^{n} a_k (x-a)^k - (x-a)^n \sum_{k=0}^{n} b_k (x-b)^k,$$

where the coefficients a_k and b_k can be obtained as linear combinations of the values $\varphi^{(k)}(a)$, $\varphi^{(k)}(b)$ for $k=0, 1, \ldots, n$. We shall denote by π the mapping $\varphi \to \varphi_0$ of $C^n(I)$ onto $C_*^n(I)$. Since the a_k, b_k are linear combinations of the $\varphi^{(k)}(a)$, $\varphi^{(k)}(b)$, it is readily seen that π is a projection, i.e. a linear mapping such that $\pi\varphi_0 = \varphi_0$ for all φ_0 in $C_*^n(I)$ and continuous. So if we set $\widetilde{u}(\varphi) = u(\pi\varphi)$, u will be a continuous linear functional on $C^n(I)$ extending u; hence there exists a distribution $\widetilde{f} \in M_n^*(I)$ such that $\widetilde{u}(\varphi) = \langle \widetilde{f}, \varphi \rangle$ and therefore, if we put $f = \rho_I \widetilde{f}$, it follows that $u(\varphi) \equiv \langle f, g \rangle$. Finally suppose $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for all $\varphi \in C_*^n(I)$, with $f = D^n \mu$, $g = D^n \upsilon$, $\upsilon, \mu \in M(I)$. Then if we put $\widetilde{f} = D^n \widetilde{\mu}$, $\widetilde{g} = D^n \widetilde{\upsilon}$, where $\widetilde{\mu}, \widetilde{\upsilon}$ are the minimal extensions of μ, υ, it follows that $\langle \widetilde{f}, \varphi \rangle = \langle \widetilde{g}, \varphi \rangle$ for all $\varphi \in C^n(I)$, so that $\widetilde{f} = \widetilde{g}$ (by theorem 6.7.8.) and therefore $f = g$. ♦

We shall now denote by $C_*^\infty(I)$ the space of all C^∞ functions φ on $I = [a, b]$ such that $\varphi^{(n)}(a) = \varphi^{(n)}(b) = 0$, for all $n \in /N_0$. Such functions can be identified with the C^∞ functions on $/R$ with support contained in I. In that space there is defined a topology making C^∞ an (F)-space by means of the sequence of norms $\|\cdot\|^n$. This being so:

6.7.10. THEOREM. *There exists a vector isomorphism $f \leftrightarrow u$ between $\mathscr{D}(I)$ and $C_*^\infty(I)'$ which is given by the formula $u(\varphi) \equiv \langle f, \varphi \rangle$. Besides*

6.7.11. $$\langle Df, \varphi\rangle = -u(\varphi'), \quad \forall \varphi \in C_*^\infty(I)$$

and, if J is a compact interval contained in I, then

6.7.12. $$u_J(\varphi) = \langle f_J, \varphi\rangle, \quad \forall \varphi \in C_*^\infty(I)$$

where u_J and f_J are the restrictions of u and f respectively to $C_*^\infty(J)$ and J.

PROOF. a) Take $f \in \mathscr{D}(I)$. Then there exists an integer n such that $f = D^n F$ with $F \in M(I)$. So if we set $u(\varphi) = \langle f, \varphi\rangle = (-1)^n \langle F, \varphi\rangle$ for all $\varphi \in C_*^\infty(I)$, u is clearly a continuous linear functional on $C_*^\infty(I)$.

b) Take $u \in C_*^\infty(I)'$. Now, a fundamental system of neighborhoods of 0 in $C_*^\infty(I)$ is given by the sets:

$$\varepsilon B_n = \{x : \|x\|^n < \varepsilon\}, \quad \varepsilon > 0, \quad n = 0, 1, \dots .$$

Hence, for any $\delta > 0$, there exists an $\varepsilon > 0$ and n such that $\varepsilon u(B_n) < \delta$. But this means that u is continuous with respect to the norm $\|\cdot\|^n$ on $C_*^\infty(I)$ and we shall see in the next paragraph that $C_*^\infty(I)$ is dense in the normed space $C_*^n(I)$. So u can be uniquely extended as a functional $\tilde{u} \in C_*^n(I)'$; i.e. there exists one and only one distribution $f \in M(I)$ such that $u(\varphi) = \langle f, \varphi\rangle$.

Finally 6.7.11. and 6.7.12. are easy consequences of the preceding results. ♦

This theorem shows that the dual space of $C_*^\infty(I)$ affords a model of the axiom system in 2.2. if we identify every function $f \in C(I)$ with the functional u such that $u(\varphi) = \int_I f\varphi$ and if we define Du by $Du(\varphi) = -u(\varphi')$.

As a matter of fact, Sobolev had first (in 1936) the idea of taking such functionals as *generalized functions* (of real variables). This method was developed in a systematic way by L. Schwartz in

1944-45. So, according to Schwartz the elements of $C_*^\infty(I)'$ are called distributions on I. The sum of two distributions u and v is the sum of the functionals u and v in the usual sense, the restriction of u to an interval $J \subset I$ is the restriction of u to $C_*^\infty(J)$, and so forth.

Till now, we have been concerned only with a compact interval J. Let us consider now an open interval Ω in $\mathbb{R}$ and let us denote by $C_*^\infty(\Omega)$ the set of all C^∞ functions on Ω with bounded carrier, contained in Ω. According to Schwartz, $C_*^\infty(\Omega)$ is provided with the topology obtained as the inductive limit of the topologies of the (F)-spaces $C_*^\infty(I)$. Then, a linear functional u on $C_*^\infty(\Omega)$ is continuous if and only if the restriction u_I is continuous. This being so, Schwartz called the elements of $C_*^\infty(\Omega)'$ **distributions** on Ω. But now theorem 6.7.10. leads directly to the following:

6.7.13. COROLLARY. *There is a vector isomorphism $f \leftrightarrow u$ between $\overline{\mathscr{D}}(\Omega)$ and $C_*^\infty(\Omega)'$.*

The result can obviously be extended to any open set Ω in $\mathbb{R}$. It must be observed however that Schwartz denotes by $\mathscr{D}(\Omega)$ the space $C_*^\infty(\Omega)$ and by $\mathscr{D}'(\Omega)$ the space of global distributions on Ω. On the other hand, Schwartz defines the topology on $\overline{\mathscr{D}}(\Omega)$ as the strong topology of $C_*^\infty(\Omega)'$. But it can be proved without difficulty that this topology is the same one that we have defined directly in chapter V, i.e. the isomorphism in 6.7.10. is a topological isomorphism.

The functional theory of distributions requires some warning in order to avoid misunderstandings. This begins already with measures. Observe, for example, that if f is a locally summable function on $\mathbb{R}$ and μ the corresponding measure, we have $\mu(I) = \int_I f(x)\,dx$ for every bounded interval I; but if we consider a one-to-one C^1 mapping h of $\mathbb{R}$ on to $\mathbb{R}$, the transformed μ^* of μ by h is given by

$$\mu^*(I) = \int_I f(h(t))h'(t)\,dt$$

so that μ^* is defined by $(f \circ h)h'$, instead of by $f \circ h$. *Hence functions and measures behave differently by change of variables, so that the identification of functions with measures works only as far as a substitution* $x = h(t)$ *with* $h' \neq 1$ *is concerned.*

The same difference arises between global distributions (as we have defined them) and distributions according to Schwartz: the first behave like functions and the second like measures, by change of variable. *In such a situation, functions cannot be identified with linear functionals, since the first are contravariant vectors and the second covariant vectors.*

As a last example, let us consider the space $\mathcal{H}^n(\Omega)$ of all functions $f \in C^n(\Omega)$ such that $f^{(k)}$ is square-summable on Ω for $k = 0, \ldots, n$ ($n \in \mathbb{N}_0$), provided with the following definition of hermitic product:

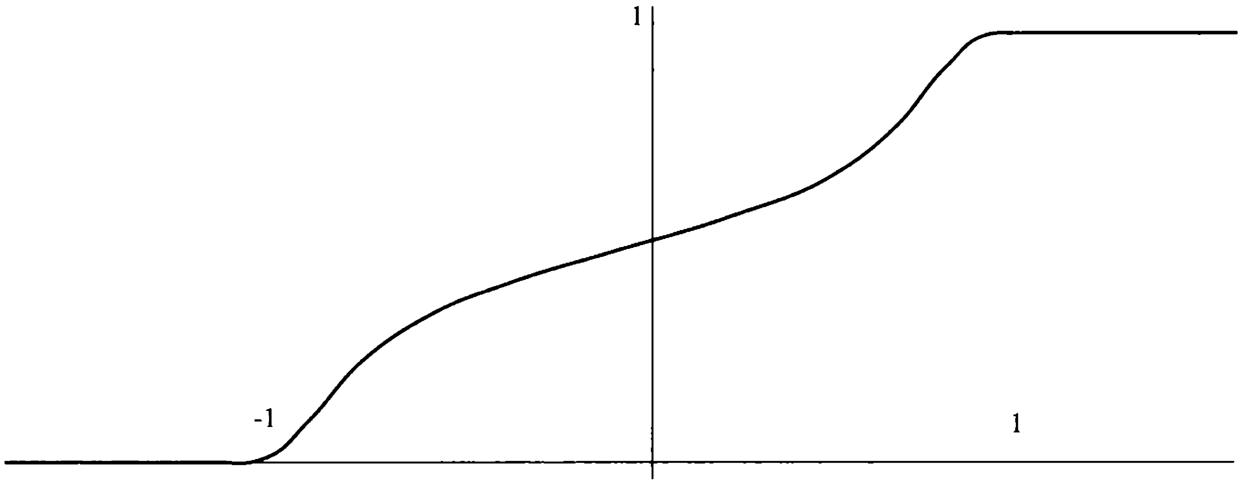
$$\langle f, g \rangle = \sum_{k=0}^{n} \int_\Omega f^{(k)} \bar{g}^{(k)}.$$

Then $\mathcal{H}^n(\Omega)$ is a Hilbert space, whose dual is just isomorphic to $\mathcal{H}^n(\Omega)$. But according to Schwartz, $\mathcal{H}^n(\Omega)'$ is identified with the space $\mathcal{H}_n(\Omega)$ of all distributions f of the form $f = \sum_{\nu=1}^{p} D^{k_\nu} f_\nu$ where p is an arbitrary integer ≥ 0 and $0 \leq k_\nu \leq n$, $f_\nu \in L^2(\Omega)$ for $\nu = 1, 2, \ldots, r$. Obviously this identification requires special care.

6.8. The approach of functions or distributions by means of C^∞ functions. Distributions according to Mikusinski

Consider the function y defined as follows:

$$y(x) = \begin{cases} \left(1 + exp \dfrac{x}{x^2 - 1}\right)^{-1} & \text{for } -1 < x < 1 \\ 0 & \text{for } \quad x \leq -1 \\ 1 & \text{for } \quad x \geq 1 \end{cases}$$



It can be seen by elementary calculations that y is a C^{∞} function on $/R$ increasing from 0 to 1. Set

6.8.1. $H_n(x)=y(nx)$ and $\delta_n=H_n'$ for $n=1, 2, \ldots$.

Then $\delta_n \in C^{\infty}$, $\delta_n(x)=0$ if $|x| \geq \frac{1}{n}$, $\delta_n(x)>0$ if $|x|<\frac{1}{n}$ and $\int_{-1}^{1} \delta_n(x)dx=1$ for $n=1, 2, \ldots$. It follows that $\delta_n \to \delta$ (cf. 5.3.).

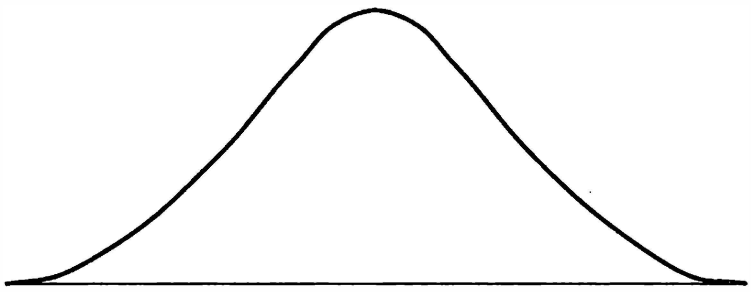
Moreover:

6.8.2. LEMMA. *If* $f \in C(/R)$ and

$$\varphi_n(x)=\int_{-\infty}^{+\infty} \delta_n(x-t)\, f(t)\, dt \quad for \quad x \in /R,\; n=1, 2, \ldots ,$$

then $\varphi_n \in C^{\infty}(/R)$ *for all n and* $\varphi_n \to f$ *uniformly on each compact interval.*

PROOF. Since $\delta_n(x-t)=0$ for $|x-t|>\frac{1}{n}$,



we have

$$\varphi_n(x)=\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} \delta_n(x-t)\, f(t)\, dt\ , \quad \forall x \in /R,\ n=1, 2, \ldots .$$

Consider now a compact interval $J=[a, b]$ and put $K=[a-1, b+1]$. Then:

$$\varphi_n(x)=\int_K \delta_n(x-t)\, f(t)\, dt\ , \quad \forall x \in J,\ n=1, 2, \ldots$$

and since $\delta_n \in C^{\infty}(/R)$ for all n, it is readily seen that $\varphi_n \in C^{\infty}(/R)$ for all n. On the other hand, by the *mean value theorem* there exists for each $x \in J$ and each $n=1, 2, \ldots$ a real ξ such that $|x-\xi|<\dfrac{1}{n}$ and

$$\varphi_n(x)=f(\xi)\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} \delta_n(x-t)\, dt=f(\xi)\,.$$

But f is *uniformly continuous* on the compact interval K. So, for every $\varepsilon>0$ there exists an integer r such that $|f(x)-f(x')|<\varepsilon$, whenever $x, x' \in K$ and $|x-x'|<\dfrac{1}{r}$. Hence $|f(x)-\varphi_n(x)|=|f(x)-f(\xi)|<\varepsilon$ for all $x \in J$ and $n>r$, which proves the lemma. ♦

6.8.3. THEOREM. *Let I be any interval, p an integer ≥ 0 and $f \in C^p(I)$. Then there exists a sequence of functions $\varphi_n \in C^{\infty}(/R)$ such that $\varphi_n^{(k)} \to f^{(k)}$ uniformly on each compact interval $J \subset I$, for $k=0, \ldots, p$.*

PROOF. a) Suppose $I = /R$. Consider the sequence φ_n defined as in the lemma. Observing that

$$D_x \delta_n(x-t) = -D_t \delta_n(x-t) \text{ and } \delta_n^{(k)}(x-t) = 0 \text{ for } |x-t| > \frac{1}{n},\ k = 0, 1, \dots,$$

it is easily seen that

$$\varphi_n^{(k)}(x) = \int_{/R} \delta_n^{(k)}(x-t)\, f(t)\, dt$$

$$= (-1)^k \int_{/R} [D_t^{(k)}\, \delta_n(x-t)]\, f(t)\, dt$$

$$= \int_{/R} \delta_n(x-t)\, f^{(k)}(t)\, dt\,, \quad \forall x \in /R,\ k = 0, 1, \dots, p,\ n = 1, 2, \dots .$$

Now, applying the lemma to the functions $f^{(k)}$, it is readily seen that $\varphi_n^{(k)} \to f^{(k)}$ uniformly on each compact interval.

b) Suppose I is closed. Then it is possible to extend f to a function $\widetilde{f} \in C^p(/R)$ (for example, if I is bounded, it is possible to make f equal to two polynomials outside I). Now, if we set

$\widetilde{\varphi}_n = \int_{/R} \delta_n(x-t)\, \widetilde{f}(t)\, dt$ and $\varphi_n = \rho_I\, \widetilde{\varphi}_n$ the theorem is proved in this case.

c) Suppose I is open. Then, there exists a one-to-one C^∞ mapping h of $/R$ onto I. So if we put $g = f \circ h$, $\psi_n = \int_{/R} \delta_n(x-t)\, g(t)\, dt$ and $\varphi_n = \psi_n \circ h^{-1}$ the theorem is proved in this case.

d) Suppose finally that I is half-open. Then it is possible to extend f as a continuous function $\widetilde{f}$ on an open interval $\widetilde{I} \supset I$, which reduces the proof to the previous case. ♦

We are going to establish a similar theorem for the space $C_*^p(I)$ but, for that purpose, it is convenient to prove, first, a lemma. It is sufficient to consider the case when I is compact, $I=[a, b]$.

6.8.4. LEMMA. *Let f be a* C^∞ *function on I such that* $f^{(k)}(a)=f^{(k)}(b)=0$ for $k=0,\dots, p$. *Then there exists a sequence of functions* $\varphi_n \in C_*^\infty(I)$ *such that* $\varphi_n^{(k)} \to f^{(k)}$ *uniformly on I, for* $k=0,\dots,p$.

PROOF. Set $\alpha_n(x)=H_n\left(x-a-\frac{1}{n}\right)-H_n\left(x-b+\frac{1}{n}\right)$, $\forall x \in IR$,

$n=1,2,\dots$, where H_n is given by 6.8.1. It is easily seen that for

$n>\frac{4}{b-a}$, we have $\alpha_n(x)=0$ outside I, $\alpha_n(x)=1$ on $\left[a+\frac{2}{n},\ b-\frac{2}{n}\right]$

and $|\alpha_n(x)| \leq 1$ for all x. On the other hand, since $H_n(x) \equiv y(nx)$, we

have $H_n^{(k)}(x)=n^k y^{(k)}(nx)$ for $k=0,\dots, n$. Then if we put $I_n=\left[a+\frac{2}{n},\ b-\frac{2}{n}\right]$

and $M=\max\limits_{-1\leq x\leq 1}(|y(x)|,\dots,|y^{(p)}(x)|)$, it is readily seen that for all

$n>\frac{4}{b-a}$ and $k=0,1,\dots, p$,

6.8.5. $$|\alpha_n^{(k)}(x)| \leq 2Mn^k \text{ on } I\backslash I_n.$$

Thus, set $\varphi_n=\alpha_n f$ for $n=1, 2, \dots$. Then $\varphi_n \in C_*^\infty$ and $\varphi_n=f$ on

I_n for $n>\frac{4}{b-a}$. On the other hand:

6.8.6. $$\varphi_n^{(k)}-f^{(k)}=\sum_{\nu=0}^{k}\binom{k}{\nu}(\alpha_n-1)^{\nu} f^{(k-\nu)}, \text{ for } k=0, 1,\dots .$$

But since $f^{(k)}(a)=f^{(k)}(b)=0$ for $k=0, 1,\dots, p$, it follows that:

$$\lim_{x\to a^+} \frac{f^{(k)}(x)}{(x-a)^{p-k}} = \lim_{x\to b^-} \frac{f^{(k)}(x)}{(x-b)^{p-k}} = 0 \text{ for } k=0,1,\ldots,p$$

and therefore there exists a sequence of numbers ε_n such that $\varepsilon_n \to 0$ and:

$$|f^{(k)}(x)| \leq \frac{\varepsilon_n}{n^{p-k}} \text{ on } I \backslash I_n, \text{ for } k=0,\ldots,p.$$

From here, from 6.8.5. and from 6.8.6., follows:

$$|\varphi_n^{(k)} - f^{(k)}| \leq 2^k(2M+1)\frac{\varepsilon_n}{n^{p-k}}, \text{ for } k=0,\ldots,p. \ \blacklozenge$$

6.8.7. THEOREM. *For every $f \in C_*^p(I)$, there exists a sequence of functions $\varphi_n \in C_*^\infty(I)$ such that φ_n converges to f in the norm $\|\cdot\|^p$.*

PROOF. Consider $f \in C_*^p(I)$. By 6.8.3., there exists a sequence of functions $\psi_n \in C^\infty(I)$ such that $\psi_n \to f$ in the normed space $C^p(I)$. We have also seen in the proof of 6.7.9. that there exists a continuous projection π of $C^p(I)$ onto $C_*^p(I)$ such that $\pi(\varphi) - \varphi$ is a polynomial for every $\varphi \in C^p(I)$. Set $\chi_n = \pi\psi_n$; then $\chi_n \in C^\infty(I)$ for all n and $\|\chi_n - f\|^p \to 0$. Finally, by the lemma, there exists for every n a sequence of functions $\chi_{n_1}, \chi_{n_2}, \ldots$, belonging to $C_*^\infty(I)$ and converging to χ_n in $\|\cdot\|^p$. Then, from the double sequence χ_{n_k}, we can select a sequence of functions $\varphi_n \in C_*^\infty(I)$ converging to f in the norm $\|\cdot\|^p$. ♦

Consider now a distribution f on $/R$ and set:

$$\varphi_n(x) = \int_{/R} \delta_n(x-t) f(t)\, dt$$

where δ_n is given by 6.8.1. Then if $f = D^n F$ with $F \in C(/R)$, it is readily seen that:

$$\varphi_n(x) = (-1)^n \int_{/R} \delta_n(x-t) F(t)\, dt$$

and from 6.8.2., it is concluded that $\varphi_n \to f$ in the distributional sense. This result can be extended to every $f \in \overline{\mathscr{D}}(/R)$ observing that on every compact interval I, f reduces to a distribution. Finally, if I is any interval in $\mathbb{R}$, we can see by the technique used in the proof of 6.8.3., that:

6.8.8. THEOREM. *For every $f \in \overline{\mathscr{D}}(I)$ there exists a sequence of functions $\varphi_n \in C^\infty(I)$ such that $\varphi_n \to f$.*

Remember that the space $\overline{\mathscr{D}}(I)$ is complete. Theorem 6.8.8. has suggested to Mikusinski a construction of the space $\overline{\mathscr{D}}(I)$ by completion of $C^\infty(I)$ with respect to the distributional topology. According to this approach, a *fundamental sequence* is a sequence of functions $\varphi_n \in C^\infty(I)$ such that, for every compact interval $J \subset I$, there exists an integer p and a sequence of functions $\Phi_n \in C^\infty(I)$ (dependent upon J) such that $\varphi_n = D^n \Phi_n$ on J and Φ_n is uniformly convergent on I. Two fundamental sequences (φ_n) and (ψ_n) are said to be equivalent if and only if for every compact interval $J \subset I$, there exists an integer p and two sequences of function Φ_n, ψ_n in $C^\infty(I)$ such that $\varphi_n = D^p \Phi_n$, $\psi_n = D^p \Psi_n$ and $\Phi_n - \Psi_n \to 0$ uniformly on J. This turns out to be actually an equivalence relation. Hence the corresponding equivalence classes are called **distributions** (i.e. global distributions according to our terminology).

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