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THEORY OF DISTRIBUTIONS*

* Este texto tem por base apontamentos coligidos por diversos alunos de José Sebastião e Silva na sequência de um curso que realizou em 1958 na Universidade de Maryland, e que posteriormente foram utilizados, e por ele revistos, na Faculdade de Ciências de Lisboa.

CHAPTER IX

FOURIER TRANSFORMATION.

9.1. Fourier transformation for tempered distributions on /*R*.

Let f be any distribution on /*R*. If the integral $\int_{/R} e^{ixy} f(y)\,dy$ is convergent on /*R*, then the distribution

9.1.1. $$g(x) = \int_{/R} e^{ixy} f(y)\,dy$$

is called the **Fourier transform** of f and we write $g(x) = \mathfrak{F}_{x|y} f(y)$, or simply $g = \mathfrak{F}f$. Frequently the Fourier transform of f is also denoted by $\hat{f}$. For simplicity we shall omit the subscript /*R* in the integral sign when no confusion can arise.

As an example, we have seen that $\int \frac{e^{ixy}}{1+y^2}\,dy = \pi e^{-|x|}$ in distributional sense. Hence,

$$\mathfrak{F}_{x|y} \frac{1}{1+y^2} = \pi e^{-|x|}.$$

From this we deduce that

$$\int e^{ixy} dy = 2\pi\delta(x),$$

and so

9.1.2. $$\mathfrak{F}1 = 2\pi\delta.$$

On the other hand, it is readily seen that

9.1.3. $$\mathfrak{F}\delta = 1.$$

We now establish some fundamental properties of the Fourier transform.

9.1.4. *If* $\mathfrak{F}f$ *and* $\mathfrak{F}g$ *exist, then* $\mathfrak{F}(\lambda f + \mu g)$ *exists for all* $\lambda, \mu \in \mathbb{C}$ and $\mathfrak{F}(\lambda f + \mu g) = \lambda(\mathfrak{F}f) + \mu(\mathfrak{F}g)$.

PROOF. This is an immediate consequence of the linearity property for integrals. ♦

9.1.5. *If* $\mathfrak{F}f$ *exists, then* $\mathfrak{F}(Df)$ *exists and* $\mathfrak{F}(Df) = -i\hat{x}(\mathfrak{F}f)$.

PROOF. $e^{ixy} f'(y) = D_y(e^{ixy} f(y)) - ixe^{ixy} f(y)$. If $\mathfrak{F}f$ exists, i.e., if $e^{ixy} f(y)$ is integrable on *IR*, then $e^{ixy} f(y) \to 0$ on *IR* as $y \to \infty$, and therefore

$$\int e^{ixy} f'(y)\, dy = -ix \int e^{ixy} f(y)\, dy. \ ♦$$

9.1.6. *If* $\mathfrak{F}f$ *exists, then* $\mathfrak{F}(\hat{y}f)$ *exists and* $\mathfrak{F}(\hat{y}f) = -iD(\mathfrak{F}f)$.

PROOF. If $\mathfrak{F}f$ exists, then by the differentiation property

$$-iD_x \int e^{ixy} f(y)\, dy = \int e^{ixy} y f(y)\, dy. \ ♦$$

Combining 9.1.4., 9.1.5. and 9.1.6., gives:

9.1.7. *If P is any polynomial, then*

$$\mathfrak{F}(P(D)f)=P(-ix)(\mathfrak{F}f)$$
$$\mathfrak{F}(P(y)f)=P(-iD)(\mathfrak{F}f).$$

We now establish some existence criteria for Fourier transforms.

9.1.8. *If f is summable on $I\!R$, then $\mathfrak{F}f$ exists and is a bounded continuous function.*

PROOF. Suppose $f\in L(I\!R)$. Since $|e^{ixy}f(y)|=|f(y)|$ for all $x, y\in I\!R$, the integral $\int|e^{ixy}f(y)|dy$ is dominated by the integral $\int|f(y)|dy$ which is convergent and independent of x. Hence, $\int e^{ixy}f(y)dy$ is uniformly convergent on $I\!R$, and thus it is convergent in the distributional sense and represents a continuous function $g(x)$ on $I\!R$. Finally,

$$|g(x)|\le\int|f(y)|dy \text{ for all } x\in I\!R. \ ♦$$

We shall denote by C_b the space of all bounded continuous functions on $I\!R$. Recall that $\breve{\mathscr{D}}$ denotes the space of all tempered distributions on $I\!R$. From 9.1.7. and 9.1.8. follows:

9.1.9. *If $f\in\breve{\mathscr{D}}$ then $\mathfrak{F}f$ exists and $\mathfrak{F}f\in\breve{\mathscr{D}}$.*

PROOF. Suppose $f\in\breve{\mathscr{D}}$. There are $m, p\in I\!N_0$ and $F\in C(I\!R)$ such that $f=D^mF$ and $F\in O(x^p)$ in the ordinary sense as $x\to\infty$.

Set $\Phi=\dfrac{F}{(1+i\hat{x})^{p+2}}$. Then $f=D^m((1+i\hat{x})^{p+2}\Phi)$, $\Phi\in C(I\!R)$, and $\Phi\in O(x^{-2})$.

Therefore, by 9.1.8., $\mathfrak{F}\Phi$ exists and $\mathfrak{F}\Phi \in C_b \subset \breve{\mathscr{D}}$. Hence, by 9.1.7., $\mathfrak{F}f$ also exists and $\mathfrak{F}f = (-i\hat{x})^m(1+D)^{p+2}(\mathfrak{F}\Phi) \in \breve{\mathscr{D}}$. ♦

We next propose to study the problems of the inversion of $\mathfrak{F}$. We observe that $\mathfrak{F}$ transformed 1 into $2\pi\delta$, δ into 1, D into multiplication by $-i\hat{x}$, and multiplication by $\hat{x}$ into $-iD$. Hence, if $\mathfrak{F}^{-1}$ exists, it must transform δ into $1/2\pi$, 1 into δ, etc. Thus we might expect that $\mathfrak{F}^{-1}$ is given by the formula:

9.1.10.
$$f(y) = \frac{1}{2\pi}\int e^{-ixy} g(x)\,dx.$$

We shall temporarily denote by $\mathfrak{F}^*$ the transformation $g \to f$ defined by 9.1.10. It is readily seen that $\mathfrak{F}^*$ has the required properties and that $\mathfrak{F}^* f$ exists for all $f \in \breve{\mathscr{D}}$. Moreover,

9.1.11. *If $f \in \breve{\mathscr{D}}$ and $g = \mathfrak{F}f$, then $f = \mathfrak{F}^* g$; conversely, if $g \in \breve{\mathscr{D}}$* and *$f = \mathfrak{F}^* g$, then $g = \mathfrak{F}f$.*

PROOF. Suppose $f \in \breve{\mathscr{D}}$ and set $g = \mathfrak{F}f$, $h = \mathfrak{F}^* g$. Then

$$h(y) = \frac{1}{2\pi}\int e^{-ixy}\left(\int e^{ixy'} f(y')\,dy'\right)dx,$$

and, if we may interchange the order of summation, we find

$$h(y) = \frac{1}{2\pi}\int\left(\int e^{ix(y'-y)}\,dx\right) f(y')\,dy'.$$

But

$$\int e^{ix(y'-y)}\,dx = 2\pi\delta(y'-y) = 2\pi\delta(y-y')$$

and by Dirac's formula

$$h(y) = \int \delta(y-y') f(y')\,dy' = f(y).$$

It is shown analogously that if $g \in \breve{\mathcal{D}}$ and $f = \mathfrak{F}^* g$, then $g = \mathfrak{F} f$. We need only justify the interchange of summations, and by 8.4.4. it is sufficient to show the convergence of the double integral

9.1.12. $$\iint e^{ix(y'-y)} f(y')\,dx\,dy'.$$

The integral

9.1.13. $$\iint \frac{e^{ix(y'-y)}}{1+x^2} f(y')\,dx\,dy'$$

with $f \in L$, is uniformly convergent on $/R$ since for all $x, y, y' \in /R$,

$$\left| \frac{e^{-ixy}}{1+x^2} \right| = \frac{1}{1+x^2} \text{ and } |e^{ixy'} f(y')| = |f(y')|,$$

and the functions $(1+x^2)^{-1}$ and $f(y')$ are summable on $/R$. Hence, by applying the operator $1-D^2$ to 9.1.13., we see that 9.1.12. is convergent for $f \in L$. The result for $f \in \breve{\mathcal{D}}$ now follows by an argument similar to the proof of 9.1.9. if we observe that 9.1.10. represents $\mathfrak{F}^* \mathfrak{F}$ and that $\mathfrak{F}^* \mathfrak{F}(Df) = Df$, $\mathfrak{F}^* \mathfrak{F}(\hat{x} f) = \hat{x} f$, for all $f \in \breve{\mathcal{D}}$. ♦

Thus we have proved that $\mathfrak{F}^* = \mathfrak{F}^{-1}$ for $\mathfrak{F}$ restricted to $\breve{\mathcal{D}}$. We ask if tempered distributions are the only distributions having a Fourier transform in the previous sense.

The answer is affirmative:

9.1.14. *If the integral* $\int e^{ixy} f(y)\,dy$ *is convergent on* $/R$, *then* $f \in \breve{\mathcal{D}}$.

PROOF. Suppose $\int e^{ixy} f(y)\,dy$ is convergent on $/R$. Then $e^{ixy} f(y)$ is of the form $(1+iy)^{-1} D_x^m D_y^n F(x, y)$ where $F(x, y) \in O(y^n)$ uniformly on each bounded interval as $y \to \infty$. Hence,

$$f(y) = (1+iy)^{-1} (e^{-ixy} D_x^m D_y^n F(x, y))$$

and, since the right member is independent of x, it follows that $f \in O(y^{2n-1})$ and so $f \in \breve{\mathscr{D}}$. ♦

The preceding results may be summarized as follows:

9.1.15. THEOREM. *$\mathfrak{F}$ is a one-to-one linear mapping of the space $\breve{\mathscr{D}}$ onto itself, changing D into multiplication by $-i\hat{x}$, multiplication by x into $-iD$, 1 into $2\pi\delta$, and δ into 1. $\mathfrak{F}^{-1}$ is given by* 9.1.10.

9.2. Fourier transformation and convolution.

The following theorem is well-known:

9.2.1. THEOREM. *If f and g are summable functions on $/R$, then $\mathfrak{F}$ transforms the convolution $f*g$ into the usual product of the continuous functions $\mathfrak{F}f$ and $\mathfrak{F}g$. That is,*

$$\mathfrak{F}(f*g)=(\mathfrak{F}f)(\mathfrak{F}g).$$

PROOF. By the theorem of Fubini-Tonelli (8.7.3.): if $f,\ g \in L$, then $f*g$ exists and $f*g \in L$. Let $\hat{f}=\mathfrak{F}f$, $\hat{g}=\mathfrak{F}g$. Then by 9.1.8., $\hat{f},\hat{g} \in C_b$ and

$$\hat{f}(x)\hat{g}(x)=\int_{/R} e^{ixu} f(u)\,du \int_{/R} e^{ixv} g(v)\,dv=\int_{/R^2} e^{ix(u+v)} f(u)g(v)\,du\,dv.$$

Now let $u+v=y$, $v=t$. Then $u=y-t$, the Jacobian of the transformation is 1 and the transformation maps $/R^2$ onto $/R^2$. Therefore,

$$\hat{f}(x)\hat{g}(x)=\int_{/R^2} e^{ixy} f(y-t)g(t)\,dy\,dt=\int_{/R} e^{ixy}\left(\int_{/R} f(y-t)g(t)\,dt\right)dy,$$

and so $\hat{f}\hat{g}=\mathfrak{F}(f*g)$. ♦

9.2.2. COROLLARY. *Let f, g be distributions on $I\!R$ of the form $f=D^mF$, $g=D^nG$, where F and G are locally summable functions satisfying the condition that there exists an integer p such that $(1+i\hat{x})^pF$ and $(1+i\hat{x})^{-p}G$ are summable on $I\!R$. Then $\mathfrak{F}(f*g)=$ $=(\mathfrak{F}f)(\mathfrak{F}g)$.*

This is a consequence of theorem 9.2.1. and properties 8.5.8. and 8.5.10. The corollary can obviously be extended to distributions which can be expressed as finite sums of the preceding forms. Recalling the definition of the space $\widehat{\mathscr{D}}$ of all rapidly decreasing distributions, it is easily deduced from 9.2.2.:

9.2.3. COROLLARY. *If $f\in\widehat{\mathscr{D}}$, $g\in\breve{\mathscr{D}}$, then $\mathfrak{F}(f*g)=(\mathfrak{F}f)(\mathfrak{F}g)$.*

In order to characterize the Fourier transforms of the rapidly decreasing distributions, we shall first establish two general criteria:

9.2.4. THEOREM. *If f is a distribution of the form D^nF, where F is a locally summable function on $I\!R$ and $F\in O(\hat{x}^{-r})$ for r, an integer ≥ 2, and if $\phi=\mathfrak{F}f$, then ϕ is a C^{r-2} function and $\phi^{(k)}\in O(\hat{x}^n)$ for $k=0$, $1,\ldots, r-2$.*

PROOF. Suppose the hypothesis is satisfied and put $\varphi=\mathfrak{F}F$. Then $\phi=(-i\hat{x})^n\varphi$ and since $\hat{x}^kF\in O(\hat{x}^{-2})$ for $k=0, 1, \ldots, r-2$, it follows that $D^k\varphi\in C_b$ for $k=0, 1,\ldots, r-2$ by 9.1.6. and 9.1.8. Hence $\phi\in C^{r-2}$ and $\phi^{(k)}\in O(\hat{x}^n)$ for $k=0, 1,\ldots, r-2$. ♦

9.2.5. THEOREM. *If ϕ is a C^r function such that $\phi^{(r)}\in O(\hat{x}^{n-r})$ for n, $r\in I\!N_0$, and if $f=\mathfrak{F}\phi$, then f is of the form $f=(1+D)^{n+2}F$ for $F\in C$ such that $F\in O(\hat{x}^{-r})$.*

PROOF. Suppose the hypothesis is satisfied and put $\varphi=(1+i\hat{x})^{-n-2}\phi$, $F=\mathfrak{F}\varphi$. Then $f=(1+D)^{n+2}F$. On the other hand, $\phi\in O(\hat{x}^{n-k})$ for $k=0, 1,\ldots, r$, and this implies $\phi^{(r)}\in O(\hat{x}^{-2})$. Hence $\hat{x}^rF\in C_b$ and so $F\in O(\hat{x}^{-r})$. ♦

9.2.6. DEFINITION. A **tempered** C^∞ function on $I\!R$ is a function

$\phi \in C^\infty(IR)$ satisfying the condition that for every $r=0, 1, \ldots$, there exists an integer n such that $\phi^{(r)} \in O(x^n)$ in the ordinary sense as $x \to \infty$. We denote by $\mathfrak{M}$ the set of all tempered C^∞ functions on IR.

It is easily seen that $\mathfrak{M}$ is a vector subspace of $\breve{\mathscr{D}} \cap C^\infty$; but observe that $\mathfrak{M} \neq \breve{\mathscr{D}} \cap C^\infty$. From 9.2.4. and 9.2.5. we have:

9.2.7. COROLLARY. *The Fourier transformation $\mathfrak{F}$ maps the convolution algebra $\widehat{\mathscr{D}}$ onto the multiplication algebra $\mathfrak{M}$.*

PROOF. a) Suppose $f \in \widehat{\mathscr{D}}$. This implies that for every $r=0, 1, 2, \ldots$, f can be represented in the form $f = \sum_{k=1}^{m} D^{r_k} F_k$ where $F_k \in O(\hat{x})^{-r-2}$ for $k=1, 2, \ldots, m$. Then if $\phi = \mathfrak{F} f$, it is easily seen from 9.2.4. that $\phi \in C^r$ and $\phi^{(r)} \in O(\hat{x}^\mu)$ where $\mu = max(r_1, \ldots, r_k, \ldots, r_m)$. Hence, $\phi \in \mathfrak{M}$.

b) Suppose $\phi \in \mathfrak{M}$. Then for every $r=0, 1, 2, \ldots$, there exists n such that $\phi^{(r)} \in O(\hat{x}^n)$. Thus if we put $f = \mathfrak{F}^* \phi$, we conclude from 9.2.5. (which obviously extends to $\mathfrak{F}^*$), that f is of the form $(1+D)^{n+2} F$, where F is a continuous function such that $F \in O(\hat{x}^{-r})$. Hence, $f \in \widehat{\mathscr{D}}$. ♦

9.3. The Fourier transformation as a continuous mapping.

It can be seen that the Fourier transformation is not continuous with respect to the topology of $\mathscr{D}$ restricted to $\breve{\mathscr{D}}$. However, we can define a stronger topology on $\breve{\mathscr{D}}$ which will make $\mathfrak{F}$ as well as D continuous, and extends the usual topologies on function subspaces of $\breve{\mathscr{D}}$. In the space C_b of all bounded continuous functions on IR a norm is usually defined by $\|f\| = \sup_{x \in IR} |f(x)|$. Then,

9.3.1. LEMMA. *$\mathfrak{F}$ defines a continuous mapping of the normed space L into the normed space C_b.*

PROOF. It is sufficient to observe that if $f \in L$, then

$\|\mathfrak{F}f\| \leq \int |f| = \|f\|_L$ (cf. proof of 9.1.8.). ♦

We shall try to define the strongest topology on $\breve{\mathscr{D}}$ making both $\mathfrak{F}$ and D continuous and inducing a topology on C_b (resp. L) weaker than the norm topology of C_b (resp. L). If such a topology exists, then $\mathfrak{F}^{-1}$ and the mapping $f \to \hat{x}f$ will also be continuous.

These considerations lead us to the following definition of convergence for sequences:

9.3.2. DEFINITION. A sequence of distributions $\{f_n\} \subset \breve{\mathscr{D}}$ converges in the **tempered sense** to a distribution $g \in \breve{\mathscr{D}}$ if there exist an integer p, a sequence of functions $\{F_n\} \subset C_b$ and a function $G \in C_b$ such that

(i) $f_n = D^p F_n$ for all n;

(ii) $g = D^p G$;

(iii) $(1+x^2)^{-p}(F_n - G)$ converges to 0 uniformly on $\mathbb{R}$ as $n \to \infty$.

It is now a simple exercise to verify that this concept of convergence satisfies all of the preceding conditions.

In order to define in $\breve{\mathscr{D}}$ the strongest topology satisfying the same condition, we shall denote by C_b^{-r} for $k = 0, 1, 2, \ldots$, the space of all distributions of the form $f = D^r(1+\hat{x}^2)^r F$ with $F \in C_b$, and we shall consider C_b^{-r} provided with the image topology of C_b by means of the mapping $F \to D^r(1+\hat{x}^2)^r F$ of C_b onto C_b^{-r}. Then it is easily seen (as in the case of distributions on a compact interval) that C_b^{-r} is a normed space and the injection $C_b^{-r} \to C_b^{-r-1}$ is compact for $r = 0, 1, 2, \ldots$. On the other hand $\breve{D} = \bigcup_{r=0}^{\infty} C_b^{-r}$ so that $\breve{\mathscr{D}}$ with the inductive limit topology of the normed spaces C_b^{-r} is an (LN^*)-space. Then it can be seen that this topology is the strongest one satisfying all preceding conditions and such that the concept of convergence for sequences agrees with that defined directly in 9.3.2.

It can also be proved that the substitution $x = t/(t^2-1)$ defines a one-to-one continuous linear mapping of the locally convex space $\breve{\mathscr{D}}$ into the locally convex space $\mathscr{D}[-1, 1]$.

9.4. Fourier transformation and scalar product.

Sometimes the Fourier transformation is defined by the formula

9.4.1.
$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{IR} e^{ixy} f(y)\,dy,$$

instead of 9.1.1. So far as no misunderstanding may arise, we shall still write in this case $g = \mathfrak{F}f$. The fundamental properties of Fourier transformations that we have previously proved are not altered by this change of form. But we now have of course,

$$\mathfrak{F}\delta = \frac{1}{\sqrt{2\pi}}, \quad \mathfrak{F}1 = \sqrt{2\pi}\,\delta.$$

The advantage of this new form is that it preserves in many cases the hermitic scalar product of two distributions.

9.4.2. DEFINITION. A **rapidly decreasing** C^∞ function on IR is a function $\phi \in C^\infty$ such that $\phi^{(n)} \in O(\hat{x}^{-r})$ for all n, $r = 0, 1, 2, \ldots$.

We shall denote by $\mathbb{S}$ the set of all rapidly decreasing functions. $\mathbb{S}$ is a proper vector subspace of $\widehat{\mathscr{D}} \cap \mathfrak{M}$. For example, $exp(-\hat{x}^2) \in \mathbb{S}$. It is easily seen that $\langle f, \phi \rangle$ exists on IR whenever $f \in \breve{\mathscr{D}}$ and $\phi \in \mathbb{S}$. Moreover, if we consider the topology on $\mathbb{S}$ defined by the sequence of norms

$$\|\phi\|_n = \sup_{x \in IR} (|\phi(x)|, (1+x^2)|\phi'(x)|, \ldots, (1+x^2)^n |\phi^{(n)}(x)|),$$

it can be proved that $\breve{\mathscr{D}}$ is isomorphic to $\mathbb{S}'$. (In the theory of Schwartz, the space $\breve{\mathscr{D}}$ is defined to be $\mathbb{S}'$). On the other hand, it is easily seen by applying 9.2.4. and 9.2.5., that $\mathfrak{F}$ maps the space $\mathbb{S}$ onto itself. Thus $\mathbb{S}$ is at the same time a multiplication algebra and a convolution algebra.

9.4.3. THEOREM. *If* $f \in \breve{\mathscr{D}}$ *and* $\phi \in \mathbb{S}$, *then* $\langle f, \phi \rangle = \langle \mathfrak{F}f, \mathfrak{F}\phi \rangle$.

PROOF. Set $g = \mathfrak{F}f$, $\psi = \mathfrak{F}\phi$. Then,

$$\langle f, \phi \rangle = \int_{I\!R} f(x)\overline{\phi(x)}dx = \int_{I\!R}\left(\frac{1}{\sqrt{2\pi}}\int_{I\!R} e^{-ixy}g(y)\,dy\right)\overline{\phi(x)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{I\!R^2} e^{-ixy}g(y)\,\overline{\phi(x)}dxdy = \int_{I\!R} g(y)\left(\frac{1}{\sqrt{2\pi}}\int_{I\!R} e^{-ixy}\overline{\phi(x)}dx\right)dy$$

$$= \int_{I\!R} g(y)\,\overline{\psi(y)}dy,$$

since the double integral exists and $exp(-ixy) = \overline{exp(ixy)}$. ♦

In the theory of Schwartz, this theorem is true by definition since $\mathfrak{F}$ is defined as the transpose of $\mathfrak{F}$ restricted to $\mathbb{S}$:

$$\langle \mathfrak{F}f, \phi \rangle = \langle f, \mathfrak{F}\phi \rangle, \text{ for all } f \in \breve{\mathscr{D}}, \ \phi \in \mathbb{S}.$$

Let us now consider the Hilbert space of all square summable functions on $I\!R$, which we shall denote by $\mathcal{H}$. We shall put

$$\|f\|_2 = \sqrt{\langle f, f \rangle} \text{ for all } f \in \mathcal{H}.$$

Convergence in this norm is called **convergence in the square mean**. It is well known that every $f \in \mathcal{H}$ can be approached in the square mean by a sequence of functions $\{\phi_n\} \subset C_*^\infty(I\!R)$, so that in particular $\mathbb{S}$ is dense in $\mathcal{H}$. On the other hand, it follows from 9.4.3. that if $\phi \in \mathbb{S}$, then $\|\phi\|_2 = \|\mathfrak{F}\phi\|_2$. Consequentely, the Fourier transformation restricted to $\mathbb{S}$ can be extended to a linear isometry of the space

$\mathcal{H}$ onto itself. We shall provisionally denote this mapping by $\widetilde{\mathfrak{F}}$. In particular, if f is a locally summable function with bounded carrier, then clearly $f\in\mathcal{H}$ and $\widetilde{\mathfrak{F}}f=\mathfrak{F}f$. Thus, in general, $\widetilde{\mathfrak{F}}f$ is given by the limit, as $a\to-\infty$ and $b\to+\infty$, in the square mean of

$$\frac{1}{\sqrt{2\pi}}\int_a^b e^{ixy}f(y)\,dy\,.$$

9.4.4. LEMMA. *If f, $g\in\mathcal{H}$ then $\int_{-n}^{n} f(x-y)g(y)\,dy$ converges uniformly on each compact subset of $I\!R$, as $n\to+\infty$, to a continuous function h such that $\mathfrak{F}h=(\widetilde{\mathfrak{F}}f)(\widetilde{\mathfrak{F}}g)$.*

PROOF. Set $f_n(x)=f(x)(H(x+2n)-(H(x-2n))$, $g_n(x)=g(x)\,(H(x+n)-H(x-n)$. Then f_n, $g_n\in L\cap\mathcal{H}$ for all n and

$\int_{-n}^{n} f(x-y)g(y)\,dy=(f_n*g_n)\,(x)$ for $|x|<n$. Hence, if we put $\hat{f}_n=\mathfrak{F}f_n$,

$\hat{g}_n=\mathfrak{F}g_n$, $\tilde{f}=\widetilde{\mathfrak{F}}f$ and $\tilde{g}=\widetilde{\mathfrak{F}}f$, we have $\hat{f}_n\,\hat{g}_n=\mathfrak{F}(f_n*g_n)\in L$ for all n, and since $\hat{f}_n\to\tilde{f}$, $\hat{g}_n\to\tilde{g}$ in the square mean, then $\hat{f}_n\hat{g}_n\to\tilde{f}\,\tilde{g}$ in the square mean, and therefore $f_n*g_n\to h=\widetilde{\mathfrak{F}}^{-1}(\tilde{f}\,\tilde{g})$ uniformly on $I\!R$.

Consequently, $\int_{-n}^{n} f(x-y)g(y)\,dy$ converges uniformly on each compact subset of $I\!R$ to the function h, which is obviously continuous. ♦

9.4.5. THEOREM. *Every function $f\in\mathcal{H}$ is a tempered distribution and $\widetilde{\mathfrak{F}}f=\mathfrak{F}f$ for every $f\in\mathcal{H}$. Moreover, convergence in the square mean implies convergence in the distributional sense and if f, $g\in\mathcal{H}$, then $f*g$ exists in the distributional sense and is a continuous function such that $\mathfrak{F}(f*g)=(\mathfrak{F}f)(\mathfrak{F}g)$.*

PROOF. Set $\tilde{f}=\mathfrak{F}f$, $\tilde{f}_0=(1+i\hat{x})^{-1}\tilde{f}$ and $f_0=\mathfrak{F}^{-1}\tilde{f}_0$, (observe that $\tilde{f}_0\in\mathcal{H}\cap L$). Since $\mathfrak{F}^{-1}(1+i\hat{x})=\delta-\delta'$, $(\delta-\delta')*f_0=(1-D)f_0$, it follows from the lemma that $f=(1-D)f_0$ and hence $f\in\breve{\mathscr{D}}$, since $f_0\in C_b$. The remainder of the theorem follows from the preceding results.

9.4.6. COROLLARY. If $f, g\in\mathcal{H}$, then $\langle f, g\rangle=\langle\mathfrak{F}f, \mathfrak{F}g\rangle$.

PROOF. It is sufficient to observe that $\mathfrak{F}$ is an isometric linear mapping of the hermitic space $\mathcal{H}$ onto itself. ♦

9.5. Fourier transformations on $/R^n$.

The Fourier transformation on $/R^n$ may be defined by

$$g(\boldsymbol{x})=\int_{/R^n} e^{i\boldsymbol{xy}}f(\boldsymbol{y})\,d\boldsymbol{y}$$

where f is a distribution on $/R^n$, and $\boldsymbol{xy}=\sum_{k=1}^{n} x_k y_k$. If the integral is convergent on $/R^n$, we write $g=\mathfrak{F}f$.

A distribution f on $/R^n$ is said to be **tempered** if and only if there exist two systems $\boldsymbol{p}$, $\boldsymbol{r}\in /N_0^n$ and a function $F\in C(/R)$ such that $f=D^{\boldsymbol{p}}F$ and $F\in O(x_1^{r_1}\dots x_n^{r_n})$ in the ordinary sense. We write $f\in\breve{\mathscr{D}}(/R^n)$ or simply $f\in\breve{\mathscr{D}}$.

All preceding properties of the Fourier transformation can be extended to the present case with the obvious modifications concerning the existence of n derivation operators and n coordinate functions $\hat{x}_1,\dots,\hat{x}_n$. Thus,

$$\mathfrak{F}(D_k f)=(-i\hat{x}_k)(\mathfrak{F}f),$$

$$\mathfrak{F}(\hat{x}_k f)=(-iD_k)(\mathfrak{F}f),$$

for all $f \in \breve{\mathscr{D}}$ and $k=1,\ldots, n$. Moreover, in the inversion formula the coefficient $\dfrac{1}{2\pi}$ must be replaced by $\dfrac{1}{(2\pi)^n}$.

Observe that if $f \in \breve{\mathscr{D}}(/R^n)$ we can define

$$g_k(\boldsymbol{x}) = \int_{/R} e^{ix_k y_k} f(\boldsymbol{x})\, dx_k,$$

the Fourier transform of f with respect to x_k. (It is easily proved that this partial integral is convergent on $\prod_{j \neq k} /R_{x_j}$). Then we write $g_k = \mathfrak{F}_k f$, and it is easily seen that

$$\mathfrak{F} = \mathfrak{F}_1 \cdots \mathfrak{F}_n .$$

For the existence of $\mathfrak{F}_k f$ it is not necessary that $f \in \breve{\mathscr{D}}(/R^n)$. It is sufficient that there exist an integer p, such that

$$f \in O(x_k^p) \text{ on } \prod_{j \neq k} /R_{x_j} \text{ as } x_k \to \infty .$$

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