JOSÉ SEBASTIÃO E SILVA

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III.1

THEORY OF DISTRIBUTIONS*

* Este texto tem por base apontamentos coligidos por diversos alunos de José Sebastião e Silva na sequência de um curso que realizou em 1958 na Universidade de Maryland, e que posteriormente foram utilizados, e por ele revistos, na Faculdade de Ciências de Lisboa.

CHAPTER IV

MULTIPLICATION AND CHANGE OF VARIABLES

4.1. Multiplication of a C^n function by a C_n distribution

As we have seen, the concept of distribution was introduced in order to render the operation D always possible, though in a formal generalized sense. But as in the case of number theory, any advantage we gain in this direction is counterbalanced by the loss of some good properties.

Multiplication of functions on the same interval is always feasible; in particular the product of two continuous functions is again a continuous function uniquely defined. But it is not possible to define the product of two completely *arbitrary* distributions, as to guarantee a minimum of properties giving some interest to such a definition.

We shall try to define the product of two distributions f and g on an interval I in $/R$, as to guarantee, *at least,* the three following conditions:

M1. *The product of two distributions f and g on I, when it exists, is again a distribution on I (which can be denoted by fg or $f \cdot g$).*
M2. *If $f, g \in C(I)$, then fg is the product of the functions f, g in the ordinary sense.*
M3. *If the product fg and $Df \cdot g$ exist, then $f \cdot Dg$ exists and $D(fg) =$ $= f \cdot Dg + Df \cdot g$.*

We shall see next that in the case when $f \in C^n$ and $g \in C_n$, the product is implicitly defined by these conditions. More precisely:

4.1.1. THEOREM. *For any integer* $n \geq 0$, *it is possible in one single way to assign to each couple* (f, g) *where* $f \in C^n(I)$ *and* $g \in C_n(I)$, *a distribution* $fg \in C_n(I)$ *not depending on n and satisfying the following conditions*:

(i) *If* $f, g \in C(I)$, *then* fg *is the product of the functions* f, g *in the ordinary sense,*

(ii) *If* $f \in C^{n+1}(I)$ *and* $g \in C_n(I)$ *then* $D(fg) = f \cdot Dg + Df \cdot g$.

By these conditions, if $f \in C^n(I)$ *and* $g = D^n G$ *with* $G \in C(I)$ *the distribution* fg *is, for each n, uniquely defined by:*

4.1.2. $$fg = f \cdot D^n G = \sum_{k=0}^{n} (-1)^k \binom{n}{k} D^{n-k}(f^{(k)}G)\,.$$

PROOF. a) We shall first prove, by induction on *n,* that in order for conditions (i) and (ii) to be satisfied, the product of $f \in C^n$ by $g \in C_n$ is *necessarily* given by 4.1.2. This statement is obviously true for $n = 0$. Suppose it is true for $n \geq 0$, we prove it is also true for $n+1$. Let $f \in C^{n+1}$, $g = D^{n+1}G$, with $G \in C$; then by condition (ii):

$$D(f \cdot D^n G) = f \cdot D^{n+1} G + Df \cdot D^n G\,.$$

Hence

4.1.3. $$f \cdot D^{n+1} G = D(f \cdot D^n G) - f' D^n G\,.$$

Now, by the induction hypothesis, we have:

$$f \cdot D^n G = \sum_{k=0}^{n} (-1)^k \binom{n}{k} D^{n-k}(f^{(k)}G)$$

and

$$f' \cdot D^n G = \sum_{k=0}^{n} (-1)^k \binom{n}{k} D^{n-k}(f^{(k+1)}G)\,.$$

Hence by substitution in 4.1.3. and by applying the well-know property $\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$ we find:

$$f\cdot D^{n+1}G=\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}D^{n+1-k}(f^{(k)}G).$$

So the statement is true for $n+1$ and consequently for all $n\geq 0$.
b) We now prove that for each *n* the product fg with $f\in C^n$ and $g\in C_n$, is uniquely defined by 4.1.2. Suppose $g=D^nG=D^nG^*$; then $G-G^*$ is a polynomial function P of degree $<n$ and:

$$f\cdot D^nG-f\cdot D^nG^*=\sum_{k=0}^{n}(-1)^k\binom{n}{k}D^{n-k}(f^{(k)}P)=f\cdot D^nP=0.$$

c) Now we prove that fg does not depend on *n*. Suppose $g=D^nG=$ $=D^{n+1}\widetilde{G}$, with $D\widetilde{G}=G$ (in the ordinary sense). Then:

$$f\cdot D^nG=\sum_{k=0}^{n}(-1)^k\binom{n}{k}D^{n-k}(f^{(k)}G)$$

and since $f^{(k)}G=f^{(k)}D\widetilde{G}=D(f^{(k)}\widetilde{G})-f^{(k+1)}\widetilde{G}$, we find as we did in a):

$$f\cdot D^nG=\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}D^{n+1-k}(f^{(k)}\widetilde{G})=f\cdot D^{n+1}\widetilde{G}.$$

d) Condition *(i)* is obviously satisfied if we define fg by 4.1.2., $n=0$. As to condition *(ii)*, it is also implied by 4.1.2., as can easily be proved by applying the property of the binomial coefficients as we did in a). ♦

Therefore, in the case $f\in C^n(I)$ *and* $g\in C_n(I)$, *the natural definition of the product is given by* 4.1.2.

The conditions formulated in 4.1.1. (equivalent to M1, M2 and M3 in this particular case) were taken as a minimal request, in order to define implicity the product in this case. This product has most of the properties of the ordinary product except that it does not exist for all couples of distributions.

4.1.4. THEOREM. *Given any integer $n \geq 0$, any two functions $\varphi, \psi \in C^n(I)$ and any two distributions f, $g \in C_n(I)$ we have:*

(j) $$(\varphi+\psi)f = \varphi f + \psi f$$

(jj) $$\varphi(f+g) = \varphi f + \varphi g$$

(jjj) $$\varphi(\psi f) = (\varphi\psi) f.$$

PROOF. For (j) and (jj) the proof is immediate. As for (jjj) observe that, to each pair of functions $\varphi, \psi \in C^n$ and to each distribution $f \in C_n$ there is assigned the product $\varphi(\psi f) \in C_n$ is such a way that:

(i) If $f \in C$, then $\varphi(\psi f)$ is the product $(\varphi\psi)f$ in the ordinary sense.

(ii) If $\varphi, \psi \in C^{n+1}$ then:

$$D[\varphi(\psi f)] = (\varphi\psi)Df + D(\varphi\psi)\cdot f.$$

By theorem 4.1.4., this is possible only if $\varphi(\psi f) = (\varphi\psi)f$. So the proof is finished. ♦

Observe that, for any $n \geq 0$, the product $\varphi\psi$ is defined, for every couple $\varphi, \psi \in C^n$ and belongs again to C^n; moreover, this operation is associative and distributive (with respect to addition) and commutative. Thus for $n = 0, 1, \ldots$, C^n is a **commutative ring**. But C^n is also a **vector space** over the field $\mathbb{C}$, and multiplication of vectors $f \in C^n$ by scalars $\lambda \in \mathbb{C}$ is related with multiplication of two vectors $f, g \in C^n$ according to the rules:

$$\lambda(fg) = (\lambda f)g = f(\lambda g).$$

All these facts can be expressed by saying that the ring C^n is a **commutative algebra** over $\mathbb{C}$.

On the contrary, C_n, for $n > 0$, is not a ring, since the product fg does not exist for all couples of distributions f, $g \in C_n$. But C_n is a complex vector space and, on the other hand, *there exists one and only one product φf for each $\varphi \in C^n$ and $f \in C_n$, with properties (j), (jj) and (jjj).* The conjunction of all these facts can be expressed by saying:

4.1.5. *For each n, $C_n(I)$ is a module over the complex algebra $C^n(I)$.*

We denote by $C^\infty(I)$, or simply C^∞, the set of all infinitely differentiable functions on I. Then C^∞ is the intersection of the C^n and it *is again a complex algebra.*

On the other hand, we have adopted the symbol $C_\infty(I)$, or simply C_∞, as an alternative notation for the set $\mathscr{D}(I)$ of all distributions on I. So C_∞ is the *union* of all vector spaces C_n.

As it follows from 4.1.5.:

4.1.6. COROLLARY. $C_\infty(I)$ is a module over the complex algebra $C^\infty(I)$.

Observe now that multiplication by complex numbers can be interpreted as a particular case of multiplication by C^∞ functions. In fact, to each $\lambda \in \mathbb{C}$ corresponds a constant function $\tilde{\lambda} \in C^\infty$, defined on *any interval I* by:

$$\tilde{\lambda}(x) = \lambda \text{ for all } x \in I.$$

It is obvious that the correspondence $\lambda \to \tilde{\lambda}$ is a one-to-one mapping of $\mathbb{C}$ onto a subset $\tilde{C}$ of C^∞ such that if $\lambda, \mu, \nu \in \mathbb{C}$, then:

$$\lambda = \mu + \nu \Leftrightarrow \tilde{\lambda} = \tilde{\mu} + \tilde{\nu}$$
$$\lambda = \mu\nu \Leftrightarrow \tilde{\lambda} = \tilde{\mu}\tilde{\nu}.$$

Moreover $\lambda f = \tilde{\lambda} f$ for every $f \in \mathscr{D}$.

Thus we can identify each number $\lambda \in \mathbb{C}$ with the corresponding function $\tilde{\lambda} \in C^\infty$ so that the field $\mathbb{C}$ becomes a subalgebra of C^∞. The number 1 *is identified with the constant function* 1 *which is the unity element of C^∞.*

4.2. Extensions of the preceding concept of product. Examples

We have seen previously (3.6) how the product of a continuous function by a measure is defined. We have defined the vector space $\mathfrak{M}_n(I)$ of *all distributions of order $\leq n$ on I.*

Then we can replace condition (i) of the theorem 4.1.1. by the stronger one:
(i') If $f \in C(I)$, $g \in \mathfrak{M}(I)$. then fg is the product of the continuous function f by the measure g as previously defined.

That being so, it is readily seen that theorems 4.1.1. and 4.1.4. can be immediately extended, by replacing $C_n(I)$ by $\mathfrak{M}_n(I)$, $C(I)$ by $\mathfrak{M}(I)$ and (i) by (i'). Thus:

4.2.1. *For each n, $\mathfrak{M}_n(I)$ is a module over the complex algebra $C^n(I)$.*

We can analogously define the product of a function f such that $f^{(n)} \in \mathfrak{M}$, by a distribution $g \in C_n$. Then $fg \in \mathfrak{M}_n$, but property (jjj) in 4.1.4. requires, in the present case, the additional assumption that $(\varphi\psi)^{(n)} \in \mathfrak{M}$ in addition to the hypothesis $\varphi^{(n)}$, $\psi^{(n)} \in \mathfrak{M}$ (which replaces the hypothesis $\varphi, \psi \in C^n$).

Another similar possibility concerns the product of a function f such that $f^{(n)} \in L^2$ by a distribution $g = D^n G$, with $G \in L^2$. Then fg is of the form $D^n \Phi$, with $\Phi \in L^1$.

Other variations can be imagined in a similar way. We have considered the product of a function by a distribution as though it were not commutative, but it is obvious that we did so only for the sake of convenience.

4.2.2. *In the preceding definitions of products, the order does not matter; i.e., the product is commutative.*

We reach another natural extension of the concept of product by trying to satisfy the following supplementary condition, which is of course satisfied in the preceding cases:

M4. *If f and g are two distributions on an interval I and if fg exists, then the product of their restrictions to every subinterval J of I exists and:*

$$\rho_J(fg) = (\rho_J f)(\rho_J g).$$

Suppose I is represented as the union of a system (I_α) of open intervals. Denote by f_α and g_α the restrictions of f and g respectively to I_α, and suppose that $f_\alpha\, g_\alpha$ exists *according to one of the preceding definitions,* (regardless of order). Then, placing $f_\alpha\, g_\alpha = h_\alpha$, one easily sees that:

1) $h_\alpha = h_\beta$ on $I_\alpha \cap I_\beta$;

2) there exists an integer υ such that the rank of h_α is less than υ for all α. Therefore by the Collecting principle (2.8.) there exists one (and only one) distribution h such that $h = h_\alpha$ on each I_α. It is natural to place $h = fg$ and it is readily seen that this new concept of product satisfies M4 as well as the preceding properties of the product holding for each interval I_α.

We can, of course, consider any open subset Ω of $/R$ instead of the interval I, and even two global distributions instead of simple distributions.

We have already seen (3.6.1.) that $f\delta = f(0)\delta$ for any continuous function f on $/R$. Suppose now that $f \in C^n(/R)$. Then by formula 4.1.2. (with $G \in \mathfrak{M}$) and 3.6.1., we obtain:

4.2.3. $$f\delta^{(n)} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f^{(k)}(0)\,\delta^{(n-k)}.$$

More generally, for any $a \in /R$:

4.2.4. $$f\delta^{(n)}(\hat{x}-a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f^{(k)}(a)\,\delta^{(n-k)}(\hat{x}-a).$$

Observe now that condition $f \in C^n(/R)$ is not necessary for the existence of $f\delta^{(n)}(\hat{x}-a)$; for that is sufficient, according to the last extension, that the *restriction of f to some neighborhood of* 0 *be a C^n function.* (This condition can even be enlarged by using the concept of value of a distribution at a point defined later on). In particular:

$$\delta^{(m)}(\hat{x}-a)\,\delta^{(n)}(\hat{x}-b) = 0 \text{ for } a \neq b,$$

and for $a=b$, this expression has no meaning according to the preceding definitions. However physicists frequently consider such products as $\delta\delta$, $\delta\delta'$, etc.

4.3. Impossibility of defining an associative multiplication for arbitrary distributions

We are going to show that:

4.3.1. *It is impossible to assign to every couple* (f, g) *of distributions a distribution* fg *as to satisfy the following:*

(i) *If* $f, g \in C(I)$ *then* fg *is the ordinary product,*

(ii) $D(fg) = Df \cdot g + f \cdot Dg\,,\ \ \forall f, g \in \mathscr{D}'(I)$,

(iii) $(fg)h = f(gh)\,,\ \ \forall f, g, h \in \mathscr{D}'(I)$,

PROOF. Suppose we have a multiplication satisfying *(i)* and *(ii),* and consider the distribution $Pf\frac{1}{x} = D\,log|x|$ (cf. 3.3.). Then

$$\left(Pf\frac{1}{x}\right)\cdot x = (D\,log|x|)x = D(x\,log|x|) - log|x| =$$
$$= D(x\,log|x|) - D(x\,log|x| - x) = 1.$$

Hence:

$$\left[\left(Pf\frac{1}{x}\right)\cdot x\right]\delta = 1\cdot\delta = \delta.$$

On the other hand, we have $x\cdot\delta = 0\cdot\delta = 0$ and therefore $\left(Pf\frac{1}{x}\right)(x\delta) = 0.$

Consequently, conditions *(i)* and *(ii)* imply $\left[\left(Pf\frac{1}{x}\right)\cdot x\right]\delta \neq$ $\neq \left(Pf\frac{1}{x}\right)(x\delta)$, which contradicts *(iii).*

This argument works for any interval I containing 0, if we consider the restrictions of $Pf\frac{1}{x}$ and δ to *I, and can be extended to any interval in /R by a suitable translation of* $Pf\frac{1}{x}$ and δ. So 4.3.1. is proved. ♦

It is clear that 4.3.1. continues to be true if we consider only the space of all distributions of the form $f=DF$, with $F\in L(I)$ instead of the space $\mathscr{D}(I)$.

It should be observed that difficulties connected with the concept of product are already found in the space $L(I)$ since the product of two locally summable functions would not be locally summable. For example, the square of the locally summable function $\frac{1}{\sqrt{x}}$ is $\frac{1}{x}$, *which corresponds to infinitely many distributions on $/R$.*

H. KÖNIG proved that it is possible to construct in infinitely many ways an extension $\widetilde{\mathscr{D}}(I)$ of $\overline{\mathscr{D}}(I)$, with the linear operator D, so that to each pair (f, g) of elements of $\overline{\mathscr{D}}(I)$, is assigned an element fg of $\widetilde{\mathscr{D}}(I)$ as to satisfy some conditions like M2, M3 and M4. However it must be observed that:

(I) *in such an extension the product of two elements of $\overline{\mathscr{D}}(I)$ is not necessarily in $\overline{\mathscr{D}}(I)$,*

(II) *the product of two elements of $\widetilde{\mathscr{D}}(I)$ does not in general exist,*

(III) *multiplication is not associative,*

(IV) *it is not possible to find, among all extensions, one that is "minimal" up to an isomorphism.*

Hence there is an irreducible indetermination in defining the product of two distributions.

4.4. Linear differential operators

Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be $n+1$ functions on O, an open set in $/R$. The linear differential operator $\sum_{j=0}^{n} \alpha_j D^j$ (of order n) is usually defined by the formula:

4.4.1. $$\left(\sum_{j=0}^{n} \alpha_j D^j\right) f(x) = \sum_{j=0}^{n} \alpha_j(x) f^{(j)}(x),$$

where f is any function having a derivative on O, in the ordinary sense, of order n. It is obvious that the same operator can be extended to any distribution f on O for which all terms exist. In particular, whenever the $\alpha_j \in C^\infty$; Then, the sum and product of two operators A and B of this form, defined by

$$(A+B)f = Af + Bf, \quad (AB)f = A(Bf), \quad \forall f \in \mathscr{D}(O),$$

are again operators of the same form. Moreover, it is easily seen that:

4.4.2. *The set Ω, of all operators of the form $\sum_{j=0}^{n} \alpha_j D^j$ with $\alpha_j \in C^\infty(O)$, is a complex non-commutative algebra.*

The nth power, A^n, of an operator $A \in \Omega$ is defined by:

$$A^0 = 1 \text{ (identity)}, \; A^n = A \cdot A^{n-1}, \text{ for } n = 1, 2, \ldots .$$

As a commutative sub-algebra of Ω, it should be mentioned the algebra Ω^* of all linear differential operators of finite order with constant coefficients, i.e., the algebra of all operators of the form

$$\sum_{j=0}^{n} \alpha_j D^j, \text{ where } \alpha_j \in \mathbb{C}.$$

As we did for functions (4.1.6.), we can identify each $\lambda \in \mathbb{C}$ with the operator λD^0, where D^0 is the identity operator, so that $\mathbb{C}$ becomes a sub algebra of Ω^*. For example:

$$(D^2+3)f = D^2 f + 3f, \quad \forall f \in \mathscr{D}$$
$$D^2 + 3 = (D + i\sqrt{3})(D - i\sqrt{3}) = (D - i\sqrt{3})(D + i\sqrt{3}), \text{ etc.}$$

If the coefficients α_j of an operator $A = \sum_{j=0}^{n} \alpha_j D^j$ are not all in $C^\infty(O)$, then A is not defined (*in any sense defined until now*) on every distribution on O. In this case, instead of the space of distributions, we can conceive other extensions of the space of continuous functions by introducing new entities (which we shall call generically **para-distributions**), as to render the operator A always defined.

This method is quite similar to the one for distributions in 2.1.

Finally, we can consider linear differential operators whose coefficients are either distributions, like δ and its derivatives, or functions which are not distributions, such as $\frac{1}{x}$, $e^{\frac{1}{x}}$, etc..

4.5. Change of variable in distributions

Let us consider a *complex*-valued function f defined on an open set $O \subset \mathbb{R}$ and a *real*-valued function h, which maps $O^* \subset \mathbb{R}$ into O. Then h assigns to each point $t \in O^*$, a point $x=h(t)$ in O; in turn, f assigns to each x the complex number $f(x)=f(h(t))$. The correspondence $t \to f(h(t))$ is a complex-valued function defined on O^* which is called the **composition** of f and h and denoted by $f \circ h$. Thus $(f \circ h)(t)=f(h(t))$.

The final operation $f \to f \circ h$ is said to be the **change of variable** (or the **substitution**) defined by h. In particular, this operation is feasible for all continuous functions f on O.

Moreover if f and h are both C^1 functions, we can apply the chain rule:

$$\frac{d}{dt} f(h(t))=f'(h(t))h'(t)$$

or

4.5.1. $$(f \circ h)'=h'(f' \circ h).$$

It should be observed that in this formula, two derivation operators are involved; one operating on functions $f \in C^1(O)$ and the other on functions $h \in C^1(O^*)$. For the sake of convenience, we denote the first by D_x and the second by D_t. Thus we can write 4.5.1. as follows:

$$D_t(f \circ h)=h'(D_x f \circ h)$$

whence, supposing $h'(t) \neq 0$ for all $t \in O^*$,

4.5.2. $$(D_x f)\circ h = \frac{1}{h'} D_t(f\circ h).$$

This formula can be expressed by saying:

4.5.3. *The change of variable defined by h transforms D_x into the differential operator* $\frac{1}{h'}D_t$.

More generally, let $f=D^nF$, $F\in C(O)$, be a distribution. By 4.5.2., we are induced to write formally:

$$(D_x^n F\circ h) = \left(\frac{1}{h'}D_t\right)^n (F\circ h).$$

Justification of this is given by the following

4.5.4. THEOREM. *If $F\in C(O)$ and $h\in C$ maps O^* into O in such a way that $\frac{1}{h'}\in C^n(O^*)$, then the expression $\left(\frac{1}{h'}D_t\right)^n(F\circ h)$ denotes a distribution in $C_n(O^*)$. Moreover, if $D_x^nF=D_x^mG$, $G\in C(O)$ we have again* $\left(\frac{1}{h'}D_t\right)^n(F\circ h)=\left(\frac{1}{h'}D_t\right)^m(G\circ h)$.

PROOF. The first part is proved by induction. The statement is obviously true for $n=0$. Suppose it is true for $n\geq 0$ and assume $\frac{1}{h'}\in C^{n+1}(O^*)$.

Then, since

$$\left(\frac{1}{h'}D_t\right)^{n+1}(F\circ h) = \frac{1}{h'}D_t\left[\left(\frac{1}{h'}D_t\right)^n(F\circ h)\right],$$

and the right side is the product of the function $\frac{1}{h'}\in C^{n+1}(O^*)$ by a distribution in $C_{n+1}(O^*)$, the statement is also true for $n+1$. Hence it is true for every integer $n\geq 0$.

The second part we can reduce to the case where O is an interval. Suppose $D_x^n F = D_x^m G$, with $m \geq n$ and place $\Phi = \mathfrak{I}^{m-n} F$; then $D^m \Phi = D^m G$ and $\Phi - G = P$ is in $\mathcal{P}_m$. By 4.5.2., we find

$$\left(\frac{1}{h'} D_t\right)^m (\Phi \circ h) = \left(\frac{1}{h'} D_t\right)^n [(D_x^{m-n} \Phi) \circ h] = \left(\frac{1}{h'} D_t\right)^n (F \circ h).$$

On the other hand:

$$\left(\frac{1}{h'} D_t\right)^m (\Phi \circ h) - \left(\frac{1}{h'} D_t\right)^m (G \circ h) = \left(\frac{1}{h'} D_t\right)^m (P \circ h) = (D_x^m P) \circ h = 0.$$

Hence

$$\left(\frac{1}{h'} D_t\right)^n (F \circ h) = \left(\frac{1}{h'} D_t\right)^m (G \circ h). \blacklozenge$$

4.5.5. DEFINITION. Under the hypothesis of theorem 4.5.4., we shall write:

$$f \circ h = \left(\frac{1}{h'} D_t\right)^n (F \circ h)$$

and we say that the distribution $f \circ h$, defined by this formula, is the **composition** of the distribution f with the function h. We sometimes use the notation $f(h(\hat{t}))$.

Now the following are easy to verify:

(i) *If $f \in C(O)$ and $h \in C(O^*)$, then $f \circ h$ is the compound of f with h in the ordinary sense.*

(ii) *If $f \in C_n(O)$ and $\frac{1}{h'} \in C^{n+1}(O^*)$, then $D_t(f \circ h) = h'(D_x f \circ h)$. (Chain rule).*

(iii) *If $f, g \in C_n(O)$, $\lambda \in \mathbb{C}$ and $\frac{1}{h'} \in C^n(O^*)$, then*

$$(f+g) \circ h = f \circ h + g \circ h \text{ and } (\lambda f) \circ h = \lambda (f \circ h).$$

(iv) *If* $f \in C_n(O)$ *and* h, k *are* C^1 *mappings of* O^* *into* O *and of* O^{**} *into* O^* *respectively, such that* $\frac{1}{h'} \in C^n(O^*)$ *and* $\frac{1}{k'} \in C^n(O^{**})$, *then* $(f \circ h) \circ k = f \circ (h \circ k)$. (Associative law).

Examples I. Translation operators are the most simple examples of change of variable. In fact the translation $\tau_a f$ of a distribution f is the distribution $f(\hat{t}-\mathrm{a})$, composition of $f(\hat{x})$ with the function $x=t-a$.

II. For any real $k \neq 0$, and $n \geq 0$, we have:

4.5.6.
$$\delta^{(n)}(kt) = \frac{1}{|k|k^n}\,\delta^{(n)}(t).$$

Indeed, putting $x=kt$, we find:

$$\delta^{(n)}(kt) = D_x^{n+1}H(x) = \frac{1}{k^{n+1}}\,D_t^{n+1}H(kt) = \frac{1}{|k|k^n}\,\delta^{(n)}(t),$$

since $H(kt)=H(t)$ or $H(kt)=1-H(t)$, depending on whether $k>0$ or $k<0$.

III. Let us see whether the change of variable defined by $x=t^2-c^2$, with $c>0$, is feasible on $\delta(x)$. The function $h(t)=t^2-c^2$ maps $/R$ into $[-c^2, +\infty[$. Also, $h'(t)=2t$, and $\frac{1}{h'(t)} = \frac{1}{2t}$ is a C^∞ function *on the open set* O *of all* $t \neq 0$ *in* $/R$. Hence the distribution $\delta(t^2-c^2)$ of t is defined on O and:

$$\delta(t^2-c^2) = D_x H(t^2-c^2) = \frac{1}{2t}\,D_t H(t^2-c^2).$$

Now it is easily seen that:

$$H(t^2-c^2) = H(t-c) + H(-t-c) = \begin{cases} 0, & \text{if } -c<t<c \\ 1, & \text{if } t<-c \text{ or } t>c \end{cases}$$

Hence:

$$\delta(t^2-c^2)=\frac{1}{2t}[D_t H(t-c)+D_t H(-t-c)]=\frac{1}{2t}[D_x H(t-c)-D_x H(-t-c)]=$$

$$=\frac{1}{2t}[\delta(t-c)-\delta(-t-c)].$$

But $\delta(-t-c)=\delta(t+c)$ (cf. 4.5.6.) and by 3.6.1.:

$$\frac{1}{2t}\delta(t-c)=\frac{1}{2c}\delta(t-c),$$

$$\frac{1}{2t}\delta(t+c)=-\frac{1}{2c}\delta(t+c).$$

Consequentely:

4.5.7. $$\delta(t^2-c^2)=\frac{1}{2c}[\delta(t-c)+\delta(t+c)] \text{ for } t\neq 0.$$

REFERENCES

[1] H. KÖNIG. *Multiplication und Variablentransformation in der Theorie der Distributionen*. Arch. Math. 6 (1955).

[2] H. KÖNIG. *Multiplication von Distributionen* I. Math. Annalen 128 (1955) 420-452. (Maths. Reviews 19-935).