JOSÉ SEBASTIÃO E SILVA

TEXTOS DIDÁCTICOS

Volume III

SERVIÇO DE EDUCAÇÃO E BOLSAS
FUNDAÇÃO CALOUSTE GULBENKIAN | LISBOA

Edição da
FUNDAÇÃO CALOUSTE GULBENKIAN
Av. de Berna — Lisboa
1999

ISBN 972-31-0971-9
Depósito Legal n.º 148 805/00

III.1

THEORY OF DISTRIBUTIONS*

* Este texto tem por base apontamentos coligidos por diversos alunos de José Sebastião e Silva na sequência de um curso que realizou em 1958 na Universidade de Maryland, e que posteriormente foram utilizados, e por ele revistos, na Faculdade de Ciências de Lisboa.

CHAPTER V

TOPOLOGIES ON SPACES OF DISTRIBUTIONS

5.1. Limit of a sequence of continuous functions

Let I be a compact interval in $/R$. Then, if f is any continuous function on I, the supremum of $|f(x)|$ on I is a non-negative real number, called the **norm** of $f \in C(I)$ and denoted by $\|f\|$.

$$\|f\| = \sup_{x \in I} |f(x)| = \max_{x \in I} |f(x)|.$$

With this definition, the vector space $C(I)$ becomes a normed space.

5.1.1. A sequence $f_0, \ldots, f_n, \ldots$ of vectors of $C(I)$ **converges** in norm to a vector g of $C(I)$ iff $\|f_n - g\| \to 0$ as $n \to \infty$. Then we write $f_n \to g$ **in norm** on I.

It is well-known that this notion of convergence is equivalent to that of "uniform convergence" on I. Remember that the sequence f_n is said to **converge uniformly** to g on I iff for every $\delta > 0$, there exists an integer N (depending on δ, but not on x), such that:

$$|f_n(x) - g(x)| < \delta, \text{ for all } n > N \text{ and all } x \in I.$$

Remember also, that uniform convergence is a sufficient condition for the limit operation to be interchangeable with the integral operator.

5.1.2. *If the sequence of continuous functions f_n converges uniformly on I to a (continuous) function g and if c is any point of I, then the sequence of (continuous) functions*

$$F_n(x)=\int_c^x f_n(\xi)\,d\xi \quad (n=1,\,2,\,\ldots;\,x\in I)$$

converges uniformly on I to the function $G(x)=\int_c^x g(\xi)\,d\xi$.

To see that, it is sufficient to apply the hypothesis observing that:

$$|F_n(x)-G(x)|\leq\int_c^x|f_n(\xi)-g(\xi)|\,d\xi\leq|I|\sup_{\xi\in I}|f_n(\xi)-g(\xi)|\leq|I|\,\|f_n-g\|<|I|\frac{\delta}{|I|},$$

if n is such that $\|f_n-g\|<\dfrac{\delta}{|I|}$, where $|I|$ denotes the lenght of I.

Thus, if we put $\Im f(x)=\int_c^x f(\xi)\,d\xi$, for any $f\in C(I)$, we can express 5.1.2. by the formula:

5.1.3. $lim(\Im f_n)=\Im(lim f_n)$, whenever $f_n\to g$ in norm (or by saying: *the operator* $\Im$ *is continuous on the normed space* $C(I)$).

On the contrary, the operator D is not continuous on the subspace $C^1(I)$ of $C(I)$, *with the same norm*. For example, consider

$f_n(x)=\dfrac{1}{n}\,sin(nx)$, for $n=1,\,2,\ldots$, then $sup|f_n(x)|=\dfrac{1}{n}$ on $/R$ for any n,

so that f_n converges to 0 uniformly on every compact subinterval I (even on $/R$); but the sequence of derivatives $f_n'(x)=cos\;nx$ does not converge uniformly (or even point-wise) on any compact interval I.

The space $\mathscr{D}(I)$ was constructed (2.2.) in order to make the operation D^n always feasible, on continuous functions. Our next purpose is to define a suitable topology on $\mathscr{D}(I)$ *so as to render the same operation continuous.*

For that purpose, we begin with the concept of convergence of distributions.

Notation. In all subjects about limits, we shall use the symbol "$\rightarrow$" as an abbreviation of "converges to", "tends to" or "approaches".

5.2. Limits of sequences of distributions

Let us consider first a *compact* interval I on $/R$. The concept of convergence for sequences of vectors in the space $\mathscr{D}(I)$, is defined so as to guarantee *at least* the two following properties:

L1. *If a sequence of elements* $f_0, \ldots, f_n, \ldots \in \mathscr{D}(I)$ *converges to an element* g *of* $\mathscr{D}(I)$ *then the sequence of derivatives* $Df_0, \ldots, Df_n, \ldots$ *converges to* Dg*; that is*:

$$f_n \rightarrow g \text{ implies } Df_n \rightarrow Dg.$$

L2. *If a sequence of functions* $f_n \in C(I)$ *converges uniformly on* I *to a function* $g \in C(I)$ *then* $f_n \rightarrow g$ *in the distributional sense (i.e. according to the new concept which we will define).*

From L1 and L2 it follows immediately that if a sequence of functions $f_n \in C(I)$ converges uniformly on I to g then $D^p f_n \rightarrow D^p g$, for any integer p.

5.2.1. DEFINITION. We say that a sequence of distributions f_n on I **converges** (or tends) to $g \in \mathscr{D}(I)$ iff there are a fixed integer p, a sequence of functions $F_n \in C(I)$ and a function $G \in C(I)$, such that:

(i) $f_n = D^p F_n$, for all n;

(ii) $g = D^p G$;

(iii) F_n converges uniformly on I to G.

It is obvious that this definition satisfies L1 and L2. Therefore, we shall see that:

5.2.2. *If $f_n \to g$ and $f_n \to g^*$, then $g = g^*$.*

In fact suppose: $\exists p, q \in \mathbb{N}_0$, sequences $F_n, F_n^* \in C(I)$ and $G, G^* \in C(I)$ such that:

(j) $f_n = D^p F_n = D^q F_n^*$, for all n;

(jj) $g = D^p G$ and $g^* = D^q G^*$;

(jjj) F_n and F_n^* converge uniformly on I, to G and G^* respectively. Assume for example $p \geq q$ and set

$$P_n = F_n - \Im^{p-q} F_n^* \text{ for } n = 0, 1, \ldots .$$

Then by (j), every $P_n \in \mathcal{P}_p$ and according to 5.1.2. and (jjj) P_n converges uniformly on I to the function $Q = G - \Im^{p-q} G^*$. Hence $Q \in \mathcal{P}_p$ and according to (jj), $g = g^*$. ♦

Remark. We have here used the well-known property: "If a sequence of polynomial functions π_n of degree $< r$ is convergent at, at least r distinct points $x_1, \ldots, x_r$, then π_n converges to a polynomial function π of degree $< r$ at every point x (even uniformly on every compact interval)". This property can be proved with the aid of the **Lagrange interpolation formula**; remember that:

$$\pi_n(x) = \sum_{k=1}^{r} \varphi_k(x) \pi_n(x_k), \text{ where}$$

$$\varphi_k(x) = \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_r)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_r)} .$$

Then by putting $c_k = \lim_n \pi_n(x_k)$ for $k=1,\ldots, r$, we have

$\lim_n \pi_n(x) = \sum_{k=1}^{r} c_k \varphi_k(x)$ for every $x \in /R$, and therefore the limit is the

polynomial $\pi(x) = \sum_{k=1}^{r} c_k \varphi_k(x)$, of degree $< r$. On the other hand, if we

put for every compact interval I, $M(I) = \max_{x,\,k} |\varphi_k(x)|$ in I, we find:

$$|\pi_n(x) - \pi(x)| \leq \sum_{k=1}^{r} M(I)\, |\pi_n(x_k) - c_k|\ , \quad \forall x \in I\ , \quad n = 1, 2, \ldots$$

and this shows that $\pi_n \rightarrow \pi$ uniformly on I.

This property can be expressed by saying: *for all integer r, the set $\mathcal{P}_r$ is closed in the normed space $C(I)$.*

5.2.3. DEFINITION. If $f_n \rightarrow g$ in $\mathscr{D}(I)$, we say that g is the **limit** of the sequence f_n and we write $g = \lim f_n$.

Observe that the uniqueness of the limit is assured by 5.2.2., which in turn is a direct consequence of definition 5.2.1.

Now it is readily seen that:

5.2.4. *If $f_n \rightarrow f^*$ and $g_n \rightarrow g^*$ in $\mathscr{D}(I)$ and if α, $\beta \in \mathbb{C}$, then $\alpha f_n + \beta g_n \rightarrow \alpha f^* + \beta g^*$; hence $\lim (\alpha f_n + \beta g_n) = \alpha \lim f_n + \beta \lim g_n$. More generally, this is also true if α and β are C^∞ functions on I.*

Let us now consider any *open* set Ω in $/R$.

5.2.5. DEFINITION. We say that a sequence f_n in $\mathscr{D}(\Omega)$ **converges** to g in $\mathscr{D}(\Omega)$ iff for every compact interval $I \in \Omega$, the sequence of distributions $\rho_I f_n$ converges to $\rho_I g$ in $\mathscr{D}(I)$ according to definition 5.2.1.

It is a simple matter to extend properties L1, L2, 5.2.2. and 5.2.4. to this new concept. For 5.2.2. apply 2.8.5. The uniqueness property enables us to write $g = \lim f_n$ iff $f_n \rightarrow g$ also in this case.

Observe that definition 5.2.5. extends immediately with the same properties, to the space $\overline{\mathscr{D}}(\Omega)$ of global distributions on Ω (cf. 2.8). For example, we can represent by

$$\sum_{n=0}^{\infty} \delta^{(n)}(\hat{x}-n) = \lim_{N\to+\infty} \sum_{n=0}^{N} \delta^{(n)}(\hat{x}-n)$$

the global distribution considered in 2.8.4.

5.3. Convergence in the mean and convergence in distributional sense. Examples.

Let us consider again a *compact interval I*. In the vector space $L(I)$, of all summable functions on *I*, the norm $\|f\|_1$ of a vector f is usually defined by:

$$\|f\|_1 = \int_I |f(x)|\, dx.$$

A sequence of vectors f_n in $L(I)$ is said to **converge in the mean** to a vector g in $L(I)$ iff $\|f_n - g\|_1 \to 0$.

5.3.1. *If a sequence of functions f_n in $L(I)$ converges almost everywhere (a.e.) in I to g and if there exists a number M such that $|f_n(x)| \le M$ for all $n=0, 1, \ldots$ and all x in I, then $g \in L(I)$ and f_n converges to g in the mean.*

This is an immediate consequence of the Lebesgue theorem.

5.3.2. *If f_n converges in the mean to g, then f_n converges to g in distributional sense.*

PROOF. Suppose that $\int_I |f_n - g| \to 0$ and put $F_n(x) = \int_c^x f_n(\xi)\, d\xi$, $G(x) = \int_c^x g(\xi)\, d\xi$, where $c \in I$. Since

$$|F_n(x)-G(x)| \le \int_c^x |f_n(\xi)-g(\xi)|\, d\xi \le \int_I |f_n-g|, \ \forall x \in I$$

it is seen that $F_n \to G$ uniformly on I. As $f_n = DF_n$ and $g = DG$, it follows that $f_n \to g$ in distributional sense. ♦

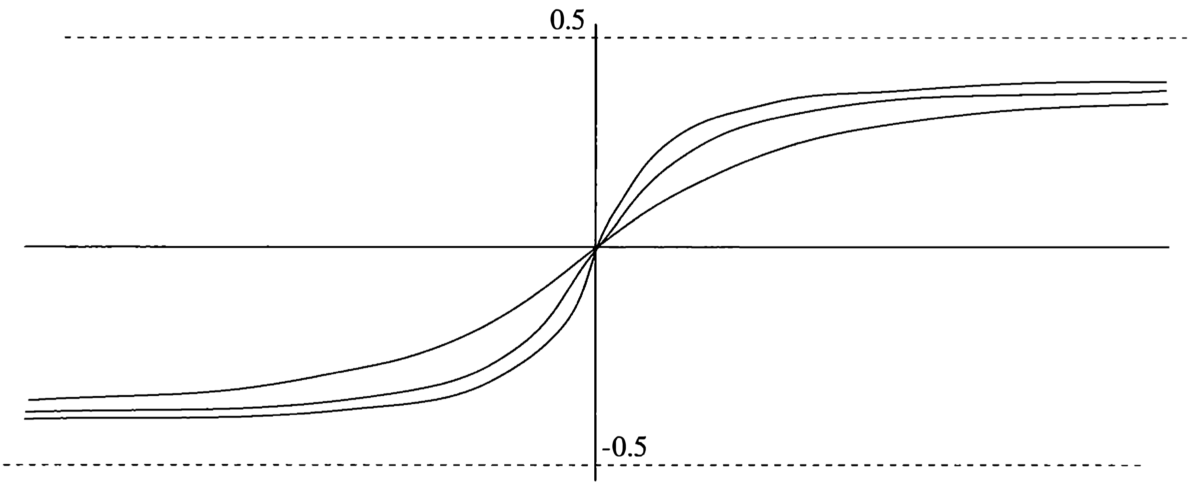
Example. Consider the sequence of functions:

$$\varphi_n(x) = \frac{1}{\pi} \frac{n}{1+(nx)^2}, \ (n=0, 1, \ldots).$$

Then, if we put $\Phi_n(x) = \int_0^x \varphi_n(\xi)\, d\xi$ for $n=0, 1, \ldots$ and all x in IR, we have $\Phi_n(x) = \left(\frac{1}{\pi}\right) arctan(nx)$ and hence:

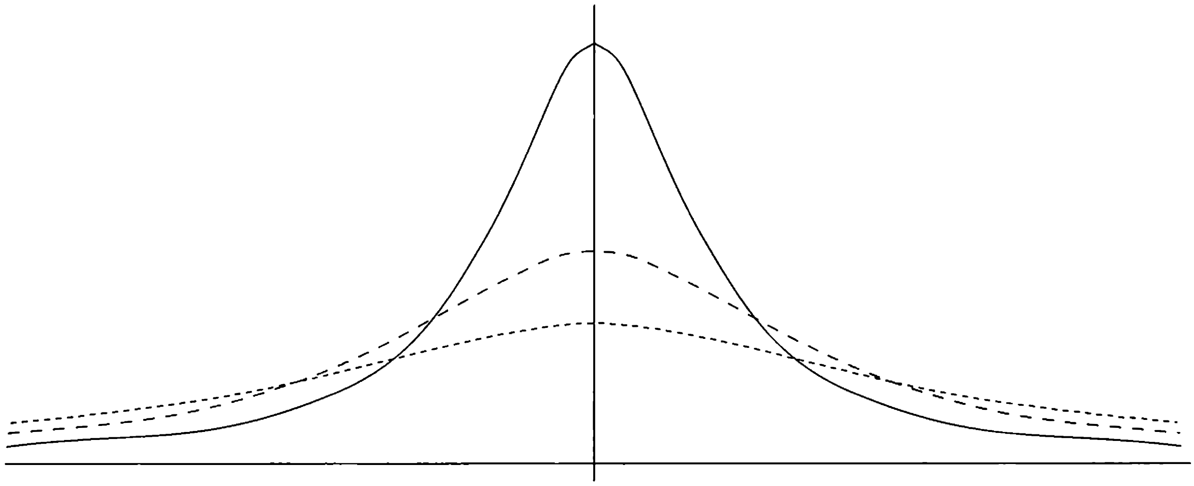
$$|\Phi_n(x)| \le \frac{1}{2}, \text{ for } n=0, 1, \ldots \text{ and any } x \in IR,$$

$$\lim_{n\to\infty} \Phi_n(x) = \begin{cases} 1/2, & \text{if } x>0 \\ 0, & \text{if } x=0 \\ -1/2, & \text{if } x<0. \end{cases}$$



So $\Phi_n(x)$ converges at every point x of IR, except zero, to the function $H(x) - \frac{1}{2}$. *Therefore*, according to 5.3.1. and 5.3.2., Φ_n

converges in the distributional sense to $H(x)-\frac{1}{2}$ *on every compact interval in /R,* and hence on /R, according to definition 5.2.5. But $\varphi_n = D\Phi_n$ for every n. Hence $\varphi_n \to D\left(H-\frac{1}{2}\right)=\delta$, that is $\delta = \lim \varphi_n$.



More generally, since φ_n is a C^∞ function on /R for each n, we have:

$$\delta^{(k)} = \lim \varphi_n^{(k)}, \quad \text{for } n=0, 1,\ldots$$

$$\varphi_n \to \delta \Rightarrow \varphi_n' \to \delta' \Rightarrow \varphi_n'' \to \delta'' \Rightarrow \cdots \Rightarrow \varphi_n^{(k)} \to \delta^{(k)}.$$

So the distribution $\delta^{(k)}$ *is expressed as the limit of a sequence of* C^∞ *functions,* $\varphi_0^{(k)},\ldots, \varphi_n^{(k)},\ldots$.

5.4. Inductive limits; (*LN**)-spaces

Till now, we have only defined a concept of convergence for sequences of distributions. In order to define in the preceding spaces of distributions a suitable topology leading to that concept of convergence (which is however sufficient for the following chapters) we need some special notions and results concerning the theory of locally convex spaces.

Consider a vector space E, a family $(E_\alpha)_{\alpha\in A}$ of vector spaces over the same field ($\mathbb{R}$ or $\mathbb{C}$) and let φ_α be, for every $\alpha\in A$, a linear mapping of E_α into E. Suppose that, on each space E_α is defined a locally convex topology τ_α (not necessarily Hausdorff). Then it is easily seen that among all locally convex topologies for which the mappings φ_α are continuous for all α, there exists one τ^* which is stronger than all the others. (A fundamental system of neighborhoods of 0 for τ^* may be the family of all circled convex and absorbing subsets $\mathcal{V}$ of E such that for every α, $\varphi_\alpha^{-1}(\mathcal{V})$ is a neighborhood of 0 for τ_α). That being so, τ^* is said to be the **inductive limit** of the topologies τ_α.

In particular, E may be the union of all E_α and φ_α the injection (identity mapping) $E_\alpha \to E$. Then $E(\tau^*)$ is said to be the inductive limit of the spaces $E_\alpha(\tau_\alpha)$. In this particular case τ^* is the strongest locally convex topology on E, inducing on each E_α, a topology weaker than τ_α.

5.4.1. DEFINITION. A locally convex space E is called a **(*LN**) space** iff E can be represented as the inductive limit of a sequence $E_1, \ldots, E_n, \ldots$ of normed spaces such that:
(1) $E_n \subset E_{n+1}$, for all n,
(2) the injection $E_n \to E_{n+1}$ is, for all n, compact (which means that all bounded sets in E_n are relatively compact with respect to the norm of E_{n+1}).
Such a sequence E_n is said to be **regular**.

5.4.2. LEMMA. *For every regular sequence (E_n) or normed spaces, there exists an increasing sequence (F_n) of Banach spaces such that:*
(i) every bounded closed set in F_n is compact in F_{n+1};
(ii) for all n, $E_n \subset F_n \subset E_{n+1}$ and the injections $E_n \to F_n$, $F_n \to E_{n+1}$, are continuous.

PROOF. Let E be the inductive limit of (E_n). For every n, denote by $\|\cdot\|_n$ the norm in E_n and set

$$B_n = \{x : \|x\|_n \le 1\}\,;\ \widetilde{B}_n = \text{closure of } B_n \text{ in } E_{n+1}\,;\ \widetilde{E}_n = \bigcup_{k=1}^{\infty} k\widetilde{B}_n .$$

As the injection $E_n \to E_{n+1}$ is continuous for all n, we can suppose that the norms $\|\cdot\|_n$ have been chosen so that $B_n \subset B_{n+1}$; i.e. $\|x\|_n \geq \|x\|_{n+1}$ for all n. Then $B_n \subset \widetilde{B}_n \subset B_{n+1}$. Since B_n is circled and convex so is $\widetilde{B}_n$, and therefore $\widetilde{E}_n$ is a vector subspace of E_{n+1}. Besides, if we place

$$g_n(x) = inf\{\rho > 0 : x \in \rho \widetilde{B}_n\}, \quad \forall x \in \widetilde{E}_n,$$

g_n will be a *norm* defined on $\widetilde{E}_n$ (since $\widetilde{B}_n \subset B_{n+1}$). Thus $F_n = \widetilde{E}_n$ becomes a normed space and (ii) follows immediately from the double inclusion $B_n \subset \widetilde{B}_n \subset B_{n+1}$ for all n. Moreover, every bounded closed set H in F_n will be compact in F_{n+1}, since there exists $\rho > 0$ such that $H \subset \rho \widetilde{B}_n$ and $\widetilde{B}_n$ is compact in E_{n+1} hence in F_{n+1} (which induces in E_{n+1} a weaker topology). It can be also proved that the spaces F_n are complete, but that is not required for the following applications. ♦

Observe that according to condition (ii), *the sequences* (E_n) *and* (F_n) *have the same inductive limit.*

In the following propositions, (E_n) denotes a regular sequence, E the inductive limit of (E_n), hence a (LN^*) space; $\|\cdot\|_n$ denotes the norm of E_n, $B_n = \{x : \|x\| \leq 1\}$. τ_n is the topology on E_n given by $\|\cdot\|_n$ and τ_∞ the topology of E (inductive limit of the topologies τ_n). *We can assume without loss of generality that* $B_n \subset B_{n+1}$ *for all n and that* B_n *is compact in* E_{n+1} (according to the lemma). That being so

5.4.3. THEOREM. *A set H is closed in E iff for every n,* $H \cap E_n$ *is* τ_n *closed.*

PROOF. a) Suppose H is closed in E. Since τ_n induces in each E_n a topology weaker than τ_∞, $H \cap E_n$ must be τ_n-closed.

b) Suppose $H \cap E_n$ is closed in E_n for all n and H is non-empty (if $H = \varnothing$, the statement is obvious). Let x_0 be any point of E such that $x_0 \notin H$. We must prove that there exists a τ_∞ neighborhood of x_0 whose intersection with H is empty. Since $x_0 \in E = \bigcup_n E_n$ and $H \neq \emptyset$, there exists at least one p such that $x_0 \in E_p$ and $H \cap E_p \neq \emptyset$. We may assume without loss of generality, that $p = 1$. Now it suffices to show that there exists an increasing sequence of circled, convex sets $U_1, U_2, \dots$ such that for all n:

(i) U_n is neighborhood of 0 in E_n,

(ii) $x_0 + U_n$ does not intersect H.

In fact, if such a sequence (U_p) exists, then $x_0 + \bigcup_1^\infty U_n$ will be a x_0-neighborhood of 0 (by the definition of inductive limit) which does not intersect H by virtue of (ii).

Since $H \cap E_1$ is closed in E_1, and $x_0 \in E_1$ we can choose a ball H_1 of center 0 in E_1, such that $x_0 + U_1$ does not intersect $H \cap E_1$. Suppose now that we have already chosen n sets $U_1, \dots, U_n$, satisfying the preceding conditions. Then $x_0 + U_n$ is compact in E_{n+1}, and *does not intersect H*. On the other hand, $H \cap E_{n+1}$ *is closed in* E_{n+1}. Therefore the distance δ_n between $x_0 + U_n$ and $H \cap E_{n+1}$, in the normed space E_{n+1} *must be* > 0. We set $V_{n+1} = \left\{ x : \|x\|_{n+1} < \frac{\delta_n}{2} \right\}$ and U_{n+1} the circled convex hull of $U_n \cup V_{n+1}$. Then $U_n \subset U_{n+1} \subset U_n \cup V_{n+1}$, so that $x_0 + U_{n+1}$ cannot intersect H, the distance between $U_n \cup V_{n+1}$ and $H \cap E_{n+1}$ being $\geq \frac{\delta_n}{2}$ in E_{n+1}. On the other hand, since U_n and V_{n+1} are bounded in E_{n+1}, so is their circled convex hull, that is U_{n+1}. Finally, as V_{n+1} is a neighborhood of 0 in E_{n+1}, so is $U_{n+1} \supset V_{n+1}$. Thus all the preceding conditions are satisfied by the sets $U_1, \dots, U_{n+1}$ and the theorem is proved by induction on n. ♦

5.4.4. COROLLARY. *Every* (LN^*) *space is a Hausdorff space.*

In fact, the theorem implies that every set reducing to a point is τ_∞-closed.

5.4.5. COROLLARY. *Let F be any topological space. Then a mapping $\varphi:E\to F$ is τ_∞ continuous iff its restriction to each E_n is a τ_n-continuous mapping of E_n into F.*

PROOF. Suppose φ is τ_∞ continuous and let M be any closed subset of F. Then $\varphi^{-1}(M)$ is τ_∞ closed and hence $\varphi^{-1}(M)\cap E_n$ is τ_n closed. The converse is analogously proved. ♦

Remark. This corollary is equivalent to the theorem itself. What the theorem means is that among *all topologies* in *E,* (not necessarily locally convex), inducing on each E_n a topology weaker than τ_n, τ_∞ is the strongest one.

5.4.6. THEOREM. *A set H is bounded in E iff there exists an integer p, such that H is contained in E_p and bounded in this normed space.*

PROOF. a) Suppose H is contained and bounded in E_p, and let V be any τ_∞ neighborhood of 0 in E. Then $V\cap E_p$ contains a τ_p-neighborhood of 0; i.e. there exists an $\varepsilon>0$ such that $\varepsilon B_p\subset V\cap E_p$. On the other hand, H being bounded in E_p implies that there exists a $\rho>0$ such that $H\subset\rho(\varepsilon B_p)$. Hence $H\subset\rho V$; i.e. H is absorbed by any τ_∞-neighborhood V of 0 and therefore is bounded in E.

b) Suppose now H is bounded in E, and put

$$C_n=\{x:\|x\|_n<n\}=n\overset{\circ}{B}_n.$$

We are going to show that H is contained in one of the open balls C_n. Suppose this is not true. Then it will be possible to take in H a sequence of points $x_1,\dots,x_n,\dots$ such that $x_n\notin C_n$ for all n. Now by a technique similar to the one used for theorem 5.4.3. we are going to prove the existence of a sequence $U_1\subset U_2\subset\cdots$ of circled convex sets such that: (i) U_k is a bounded and closed neighborhood of 0 in E_k contained in $\overset{\circ}{B}_k$ for all k; (ii) $\frac{1}{n}x_n\notin U_k$ for all k and n. For example, take $U_1=\frac{1}{2}B_1$; since $x_n\notin C_n=n\overset{\circ}{B}_n$ and $B_1\subset B_n$ for all n, then $\frac{1}{n}x_n\notin U_1$ for all n. Suppose that we have already chosen k sets

$U_1, \ldots, U_k$ satisfying the preceding conditions and place:

$$M_k = \left\{x_1, \frac{1}{2}x_2, \ldots, \frac{1}{k}x_k\right\} \cup \left(E_{k+1} \setminus \overset{\circ}{B}_{k+1}\right).$$

Then M_k is a *closed set* in E_{k+1} which contains all points $\frac{1}{n}x_n$ and *does not intersect* U_k, according to (i) and (ii). On the other hand, U_k is compact in E_{k+1}. Hence, if we set $\delta_k = dist(U_k, M_k)$, $V_k = \left\{x : \|x\|_k < \frac{\delta_k}{2}\right\}$ and U_{k+1} = circled convex hull of $U_k \cup V_k$, we can prove, as in theorem 5.4.3., that U_{k+1} is a circled convex hull compact in E_{k+2} contained in $\overset{\circ}{B}_{k+1}$, and it is obvious that $\frac{1}{n}x_n \notin U_{k+1}$ for all *n*. Thus the existence of a sequence satisfying (i) and (ii) is now proved.

Now it is readily seen that the set $V = \bigcup_1^\infty U_n$ is a τ_∞-neighborhood of 0 such that $\frac{1}{n}x_n \notin V$. But then, for every $\rho > 0$, we should have $x_n \notin \rho U$ for all $n > \rho$ and this is impossible, the set *H* being bounded in *E*. Consequently, there exists at least one *p* such that $H \subset C_p$, which implies that *H* is contained and bounded in E_p. ♦

5.4.7. COROLLARY. *A sequence* (x_n) *of points of E converges to a point x of E if and only if there exists a p such that all the points* x_n *and x belong to* E_p *and* $x_n \to x$ *in* E_p.

PROOF. a) Suppose there exists *p* such that $x_n \to x$ in E_p. Then, since τ_∞ induces on E_p a topology weaker than τ_p, $x_n \to x$ in *E*.

b) Suppose $x_n \to x$ in *E*. Then the set $X = \{x_n\}_{n \in IN} \cup \{x\}$ is bounded in *E* and so there exists an *r* such that *X* is bounded in E_r. Hence the aderence $\overline{X}$ of *X* in E_{r+1} is compact and, according to a general theorem of Topology, τ_∞ *induces* on $\overline{X}$ the same topology as does τ_{r+1}. So, as $x_n \to x$ in *E*, we can conclude that $x_n \to x$ in E_{r+1}. ♦

5.4.8. COROLLARY. *If E is infinite dimensional, then E is not metrisable.*

PROOF. Suppose E is metrisable. Then there exists a fundamental system $\mathcal{V}$ of neighborhoods of 0 in E which is countable: $\mathcal{V}=\{V_1,\dots, V_n,\dots\}$. Now at least one of these neighborhoods must be bounded in E; otherwise there would exist, for each n, an $x_n \in V_n$ such that $x_n \notin C_n = n\overset{\circ}{B}$ (by the theorem) and thus (x_n) would be a sequence *converging to* 0 *and unbounded*, which is impossible. Let V_p be a neighborhood of the system $\mathcal{V}$ which is bounded in E, then V_p is bounded in E_m for some m and hence relatively compact in E_{m+1}. As V_p is also a neighborhood of 0 in E_{m+1}, it follows that E_{m+1} is finite dimensional and $E_{m+1}=E_{m+2}=\cdots=E$ since V_p is a neighborhood of 0 in E which implies $E=\bigcup_{1}^{\infty} kV_p$. ♦

5.4.9. COROLLARY. *Every* (LN^*) *space E is an* (M)*-space (i.e. a Montel space where every bounded set is relatively compact).*

In fact if M is a bounded set in E, then M is bounded in some E_p and hence relatively compact in E_{p+1}. Since the Hausdorff topology τ_∞ induces on E_{p+1} a topology weaker than τ_{p+1}, it follows that M is also relatively compact in E.

5.4.10. COROLLARY. *Every* (LN^*) *space E is reflexive (i.e. the strong bidual E″ of E is topologically isomorphic to E).*

In fact, E being the inductive limit of a family of normed spaces is a *barreled* space, and this along with 5.4.9., implies that E is reflexive.

5.4.11. COROLLARY. *Every* (LN^*) *space is complete.*

PROOF. By the theorem there exists a sequence (C_n) of bounded sets in E such that every bounded set H in E is contained in one of the C_n. This implies that the polar sets $\overset{\circ}{C}_1, \overset{\circ}{C}_2,\dots$ form a fundamental sequence of neighborhoods of 0 in E' which is *countable*. Hence

E' is metrisable and as E is isomorphic to E'', it follows that E is complete. ♦

Remark. The (LN^*) spaces turn out to be "Schwartz spaces", according to the terminology of Grothendieck. They can be characterized as the *strong duals of the Schwartz metrisable spaces.* Observe, however that for a direct definition of (LN^*) spaces, as well as for their application to define *directly* the topology in spaces of distributions, the preceding theorems are needed, the results of Grothendieck being insufficient. For further information, see the references at the end of the chapter.

5.5. Topology of $\mathscr{D}(I)$, when I is a compact interval

In order to arrive at our goal, we still need a criterium concerning a particular case of the concept of inductive limit introduced at the beginning of 5.4. Let E be a normed space, F a vector space (over the same field $/R$ or $\mathbb{C}$), and φ a linear mapping of E onto F. It is easily seen that the strongest topology on F making φ continuous (the so called **image top** of the topology of E by φ) can be defined by the semi-norm corresponding to the set $\varphi(B)$ where B is the unit ball in E. Then, *F becomes a seminormed space, which is a normed space iff the kernel $N(=\varphi^{-1}(0))$ of φ is closed in E (which is a necessary and sufficient condition for the set $\{0\}$ to be closed in F).*

This being so, we let I be a compact interval on $/R$. Then $C(I)$ is a normed space according to the usual definition of norm recalled in 5.1. On the other hand, we have seen that the operator D^n for $n=1, \ldots$ defines a linear mapping of $C(I)$ *onto* the space $C_n(I)$ of distributions of rank $\leq n$. In these circumstances it is natural to consider the space $C_n(I)$ provided with the image topology τ_n of the topology τ of $C(I)$ by means of D^n. Now, the kernel of D^n (i.e. the set of all functions φ such that $D^n\varphi=0$ in C_n) is the set $\mathscr{P}_n$, which as we have seen *is closed* in $C(I)$ (cf. remark to theorem 5.2.2.). Consequently:

5.5.1. *The vector space $C_n(I)$ with the topology τ_n is a normed space.*

Observe that now we have, both topologically and algebraically:

$$C_n(I) \cong C(I)/\mathcal{P}_n .$$

Besides, it is easily seen that:

5.5.2. *A sequence* (f_k) *of distributions on* I *converges to a distribution* g *on* I *in the* τ_n *topology iff exists a sequence of functions* F_k *in* $C(I)$ *and a function* $G \in C(I)$ *such that* $f_k = D^n F_k$ *for all* k, $g = D^n G$ *and* $F_k \to G$ *uniformly on* I.

Now, we are going to prove that:

5.5.3. *The normed spaces* $C_n(I)$, $n=1, 2, \ldots$ *form a regular sequence according to definition* 5.5.1.

PROOF. Remember that the unit ball B_n in $C_n(I)$, is the image of the unit ball B in $C(I)$; that is

$$B_n = D^n B = \{f : f = D^n F,\ F \in C(I),\ \|F\| \leq 1\}.$$

Now set

$$B' = \left\{\Phi : \Phi(x) \equiv \int_c^x F(\xi)\, d\xi,\ F \in B\right\} \quad (c \in I).$$

Then for all $\Phi \in B'$ and all x, $x+h \in I$,

$$|\Phi(x+h) - \Phi(x)| = \left| \int_x^{x+h} F(\xi)\, d\xi \right| \leq |h|.$$

This shows that B' is *equicontinuous* on the (compact) interval l; B' is also bounded, of course. Hence, according to Ascoli's theorem, B' is *relatively compact* in $C(I)$. But D^{n+1} defines a continuous mapping of $C(I)$ onto $C_{n+1}(I)$. Consequentely, the set $B_n = D^n B = D^{n+1} B'$ is *relatively compact in* $C_{n+1}(I)$, for all n, and this implies that the sequence $(C_n(I))$ is regular. ♦

Remember now that:

$$\mathscr{D}(I)=C_\infty(I)=\bigcup_{n=1}^{\infty} C_n(I);$$

thus it is natural to consider the vector space $\mathscr{D}(I)$ provided with the topology C_∞, which is the inductive limit of the topologies of the normed spaces $C_n(I)$.
Then, according to 5.5.3.:

5.5.4. $\mathscr{D}(I)$ *is a* (LN^*) *space.*

In particular from 5.4.7. and 5.5.2.:

5.5.5. *The concept of convergence for sequences in the locally convex spaces* $\mathscr{D}(I)$ *is the same as the introduced directly in* 5.2.1.

5.6. Topology of $\mathscr{D}(\Omega)$, where Ω is an open set

For this case, we need the concept of **projective limit**.

Consider a vector space E, a family $(F_\alpha)_{\alpha\in A}$ of vector spaces over the same field ($/R$ or $\mathbb{C}$) and let φ_α be, for each $\alpha\in A$, a linear mapping of E onto F_α. Suppose that, on each F_α there is defined a locally convex topology τ_α. Then it is easily seen that among all locally convex topologies on E for which each φ_α is continuous, there is one τ^* weaker than all the others: this is called the **projective limit** of the topologies τ_α in E, with respect to the mapping φ_α. To define τ^* directly it is sufficient to observe the following:

5.6.1. *A filter* $\mathfrak{F}$ *converges to* 0 *in* $E(\tau^*)$ *if and only if the filter* $\varphi_\alpha(\mathfrak{F})$ *converges to* 0 *in* $F_\alpha(\tau_\alpha)$ *for each* $\alpha\in A$.

Now let Ω be any open set in $/R$. For any compact interval $I\subset\Omega$, the restriction operator P_I is a linear mapping of $\mathscr{D}(\Omega)$ onto $\mathscr{D}(I)$. But $\mathscr{D}(I)$ has been defined as a locally convex space, as a (LN^*)-space.

Hence, *it is natural to consider the vector space* $\mathscr{D}(\Omega)$ *provided with the projective limit of the topologies of the spaces* $\mathscr{D}(I)$ *with respect to the operators* P_I.

Then according to 5.6.1.:

5.6.2. *The concept of convergence for sequences in locally convex spaces* $\mathscr{D}(\Omega)$ *is the same as introduced in* 5.2.5.

It must be observed however that the locally convex space $\mathscr{D}(\Omega)$ is not complete. For example, the sequence of distributions $f_n = \sum_1^n \delta^{(k)}(\hat{x}-k)$ on $/R$ is a Cauchy sequence on each compact interval I, hence on $/R$, but it does not converge to a distribution on $/R$.

Obviously, we can define on the space $\overline{\mathscr{D}}(\Omega)$ of all global distributions on Ω a topology as we did for $\mathscr{D}(\Omega)$. Then it is readily seen that $\mathscr{D}(\Omega)$ *is a locally convex subspace of* $\overline{\mathscr{D}}(\Omega)$*, which is dense in* $\overline{\mathscr{D}}(\Omega)$. Moreover, remembering that every space $\mathscr{D}(I)$ (I a compact interval) is complete being a (LN*)-space, is easily shown that:

5.6.3. $\overline{\mathscr{D}}(\Omega)$ *is complete.*

So $\overline{\mathscr{D}}(\Omega)$ can also be obtained by the completion of $\mathscr{D}(\Omega)$.

Finally, it can be proved that the preceding topologies on $\mathscr{D}(I)$ and $\mathscr{D}(\Omega)$ coincide with the strong topologies introduced by L. Schwartz in these spaces, considered as the duals of certain spaces of C^∞ functions.

Remark. The (LN*)-spaces turn out to be a special category of Schwartz spaces; they can be characterized as the strong duals of Schwartz metrisable spaces. This category of spaces has been pointed out by the author in 1952, first for the study of spaces of analytic functions (see references below). But its application to spaces of distributions required some developments introduced in 1954. The

(*LN**)-spaces occur in a great number of situations in functional analysis, as far as distributions and analytic functions are concerned. The specific properties of these spaces are not implied in the paper by Grothendieck, on (*F*) and (*DF*) spaces, where the Schwartz spaces were introduced. Some research on (*LN**)-spaces has been made by Yoshinaga and a generalization of this class of spaces has been presented by Kaikov.

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